Novel Sampling Formulas Associated with Quaternionic Prolate Spheroidal Wave functions

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Abstract. The Whittaker–Shannon–Kotel’nikov (WSK) sampling theorem provides a reconstruction formula for the bandlimited signals. In this paper, a novel kind of the WSK sampling theorem is established by using the theory of quaternion reproducing kernel Hilbert spaces. This generalization is employed to obtain the novel sampling formulas for the bandlimited quaternion-valued signals. A special case of our result is to show that the 2D generalized prolate spheroidal wave signals obtained by Slepian can be used to achieve a sampling series of cube-bandlimited signals. The solutions of energy concentration problems in quaternion Fourier transform are also investigated.

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1. Introduction

The sampling theory, as one of the basic and fascinating topics in engineering sciences, is crucial for reconstructing the continuous signals from the information collected at a series of discrete points without aliasing because it bridges the continuous physical signals and the discrete domain. After the celebrated Whittaker Shannon Kotel’nikov (WSK) sampling theorem established, there have been numerous proposals in the literature to generalize the classical WSK sampling expansions in various areas. The goal of this paper is to use the theory of quaternion reproducing kernel Hilbert method to
obtain a generalization of WSK sampling for a general class of bandlimited quaternion-valued signals.

On the other hand, special functions [4] such as Hermite and Laguerre functions have played an important role in classical analysis and mathematical physics. In a series of papers, Slepian et al. [15–17,22] extensively investigated the remarkable properties of the prolate spheroidal wave functions (PSWFs) which are a class of special functions. For fixed $\tau$ and $\sigma$, the PSWFs of degree $n$ denoted by $\varphi_n$ constitute an orthogonal basis of the space of $\sigma$-bandlimited signals with finite energy, that is, for continuous finite energy signals whose Fourier transforms have support in $[-\sigma, \sigma]$. They are also maximally concentrated on the interval $[-\tau, \tau]$ and depend on parameters $\tau$ and $\sigma$. PSWFs are characterized as the eigenfunctions of an integral operator with kernel arising from the sinc functions $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$:

$$\frac{\sigma}{\pi} \int_{-\tau}^{\tau} \varphi_n(x) \text{sinc} \left( \frac{\sigma}{\pi} (y - x) \right) dx = \mu_n \varphi_n(y), \ |y| \leq \tau. \quad (1.1)$$

It has been shown that (1.1) has solutions in $L^2([-\tau, \tau])$ only for a discrete set of real positive values of $\mu_n$ say $\mu_1 > \mu_2 > \cdots$ and that $\lim_{n \to \infty} \mu_n = 0$. The corresponding solutions, or eigenfunctions, $\varphi_1(y), \varphi_2(y), \ldots$ can be chosen to be real and orthogonal on $(-\tau, \tau)$.

The variational problem that led to (1.1) only requires that equation to hold for $|y| \leq \tau$. With $\varphi_n(x)$ on the left-hand side of (1.1) gives for $|x| \leq \tau$, however, the left is well defined for all $y$. We use this to extend the range of definition of the $\varphi_n$’s and so define

$$\varphi_n(y) := \frac{\sigma}{\pi \mu_n} \int_{-\tau}^{\tau} \varphi_n(x) \text{sinc} \left( \frac{\sigma}{\pi} (y - x) \right) dx, \ |y| \geq \tau.$$

The eigenfunctions $\varphi_n$ are now defined for all $y$. This leads to a dual orthogonality

$$\int_{-\tau}^{\tau} \varphi_n(x) \varphi_m(x) dx = \mu_n \delta_{mn}, \quad (1.2)$$

$$\int_{-\infty}^{\infty} \varphi_n(x) \varphi_m(x) dx = \delta_{mn}. \quad (1.3)$$

In [24], Zayed proved that there are other systems of functions possess similar properties to those of prolate spheroidal wave functions. Moumni and Zayed [18] then extended the results to the higher dimension and derived a novel sampling formula for general class of bandlimited functions. This sampling formula [18] is a generalization of Walter and Shen’s result [23] on sampling with the PSWFs. In the present paper, we study the quaternionic prolate spheroidal wave functions (QPSWFs), which refine and extend the PSWFs. The QPSWFs are ideally suited to study certain questions regarding the relationship between quaternion-valued signals and their Fourier transforms. We illustrate how to apply the QPSWFs for the quaternion Fourier transform to analyze Slepian’s energy concentration problem and sampling theory. We address all the above issues and explore some basic facts of the arising quaternion-valued function theory.