Notes on von Neumann–Jordan and James Constants for Absolute Norms on \( \mathbb{R}^2 \)

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Abstract. Let \( \| \cdot \|_\psi \) be the absolute norm on \( \mathbb{R}^2 \) corresponding to a convex function \( \psi \) on \([0, 1]\) and \( C_{NJ}(\| \cdot \|_\psi) \) its von Neumann–Jordan constant. It is known that \( \max\{M_1^2, M_2^2\} \leq C_{NJ}(\| \cdot \|_\psi) \leq M_1^2 M_2^2 \), where \( M_1 = \max_{0 \leq t \leq 1} \psi(t)/\psi_2(t) \), \( M_2 = \max_{0 \leq t \leq 1} \psi_2(t)/\psi(t) \) and \( \psi_2 \) is the corresponding function to the \( \ell_2 \)-norm. In this paper, we shall present a necessary and sufficient condition for the above right side inequality to attain equality. A corollary, which is valid for the complex case, will cover a couple of previous results. Similar results for the James constant will be presented.

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1. Introduction and Preliminaries

Among various geometric constants of Banach spaces \( X \), the von Neumann–Jordan constant \( C_{NJ}(X) \) and the James constant \( J(X) \) are most widely treated (see e.g., [3–5,9,10] for some basic results). In the paper Saito et al. [7] a sequence of results on the von Neumann–Jordan constant for absolute norms on \( \mathbb{C}^2 \) was presented. In particular they showed the following ([7, Theorem 2]): Let \( \| \cdot \|_\psi \) be the absolute normalized norm on \( \mathbb{C}^2 \) corresponding to a convex function \( \psi \) satisfying \( \max\{1 - t, t\} \leq \psi(t) \leq 1 \) on \([0, 1]\). Let \( M_1 = \max_{0 \leq t \leq 1} \psi(t)/\psi_2(t) \) and \( M_2 = \max_{0 \leq t \leq 1} \psi_2(t)/\psi(t) \), where \( \psi_2 \) is the convex function corresponding to the \( \ell_2 \)-norm. Then

\[
\max\{M_1^2, M_2^2\} \leq C_{NJ}(\| \cdot \|_\psi) \leq M_1^2 M_2^2. \tag{1}
\]

For the right side inequality the following was shown.

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Theorem A (cf. [7, Theorem 3]). Let $\psi \in \Psi$ and let $\psi(t) = \psi(1-t)$ for all $0 \leq t \leq 1$. Assume that $M_1 = \max_{0 \leq t \leq 1} \psi(t)/\psi_2(t)$ or $M_2 = \max_{0 \leq t \leq 1} \psi_2(t)/\psi(t)$ is taken at $t = 1/2$. Then $C_{NJ}(\| \cdot \|) = M_1^2 M_2^2$.

Under the same condition in Theorem A, they showed that $\max\{M_1, M_2\} = C_{NJ}(\| \cdot \|)$ if and only if $\psi \geq \psi_2$ or $\psi \leq \psi_2$ ([7, Theorem 3]).

In this paper, we shall present a necessary and sufficient condition for the complex case and includes Theorem A and Theorem B below.

Theorem B ([7, Theorem 1]). Let $\psi \in \Psi$.

(i) Let $\psi \geq \psi_2$. Then $C_{NJ}(\| \cdot \|) = \max_{0 \leq t \leq 1} \psi(t)^2/\psi_2(t)^2$.
(ii) Let $\psi \leq \psi_2$. Then $C_{NJ}(\| \cdot \|) = \max_{0 \leq t \leq 1} \psi_2(t)^2/\psi(t)^2$.

Next, we shall discuss the James constant $J(\| \cdot \|)$. Owing to the inequality $C_{NJ}(X) \leq J(X)$ shown recently in [10] (cf. [11, 12]), it will be readily obtained that

$$\max\{M_1^2, M_2^2\} \leq J(\| \cdot \|) \leq \sqrt{2} M_1 M_2. \quad (2)$$

We shall characterize equality-attainedness for the both sides inequalities in (2), from which a couple of previous results by Mitani and Saito [6] will be derived as a corollary. Finally, we shall present an example, which will illustrate that our results provide a new way to determine these constants, where some related geometric constants such as the modified von Neumann–Jordan and Zbăganu constants are also treated.

A norm $\| \cdot \|$ on $\mathbb{C}^2$ is called absolute provided $\|(z, w)\| = ||(z, |w|)||$ for all $(z, w) \in \mathbb{C}^2$, and normalized provided $\|(1, 0)\| = \|(0, 1)\| = 1$. Let $\Psi$ be the collection of all convex functions $\psi$ on $[0, 1]$ satisfying the condition

$$\max\{1-t, t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1). \quad (3)$$

For any $\psi \in \Psi$ define

$$\|(z, w)\|_\psi = \begin{cases} \langle |z| + |w| \rangle \psi \left( \frac{|w|}{|z| + |w|} \right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases} \quad (4)$$

Then, $\| \cdot \|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ and satisfies $\psi(t) = \|(1-t, t)\|_\psi$; and any absolute normalized norm on $\mathbb{C}^2$ is obtained in this way ([2]). For the $\ell_p$-norm $\| \cdot \|_p$ the corresponding convex function $\psi_p$ is given by

$$\psi_p(t) = \begin{cases} (1-t)^p + t^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases}$$

The von Neumann–Jordan constant for a Banach space $X$ is defined to be

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : (x, y) \neq (0, 0) \right\}.$$