Periodic structure of $\sigma$-permutation maps II. The case $\mathbb{T}^n$

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Summary. In this paper we extend the periodic structure obtained for $\sigma$-permutation maps in the $n$-dimensional cube $I^n$, $I = [0,1]$ (see [BL]). Now we wonder whether there exists a similar periodic structure for $\sigma$-permutation maps defined on the $n$-dimensional torus $\mathbb{T}^n$.


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1. Introduction

We have already obtained the periodic structure of $\sigma$-permutation maps on the $n$-dimensional cube $I^n$, $I = [0,1]$ (see [BL]). Now we wonder whether there exists a similar periodic structure for $\sigma$-permutation maps defined on the $n$-dimensional torus $\mathbb{T}^n$.

$\mathbb{T}^n = S^1 \times \cdots \times S^1$, $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$.

Given a nonempty metric space $X$, the map $F : X^n \to X^n$ is a $\sigma$-permutation map if

$$F(x_1, x_2, \ldots, x_n) = (f_{\sigma(1)}(x_{\sigma(1)}), \ldots, f_{\sigma(n-1)}(x_{\sigma(n-1)}), f_{\sigma(n)}(x_{\sigma(n)})),$$

where $x_i \in X$, $f_i : X \to X$ is a continuous map for $i = 1, \ldots, n$, and $\sigma$ is a cyclic permutation of the elements $1, 2, \ldots, n$.

We have found that the periodic structure of $F$ depends strongly on the periodic structure of the map $g : X \to X$ where $g = f_{\sigma(1)} \circ \cdots \circ f_{\sigma(n)}(1)$. In the case $X = S^1$ the periodic structure of continuous maps $g : S^1 \to S^1$ is well known (see the monograph [ALM]) and depends in turn on the degree of the map.

The paper is organized as follows. First, we introduce the basic notation and terminology used throughout the paper. Next we state auxiliary results on iterates of $\sigma$-permutation maps and the connection between their periodic structure and that of some related maps. Then we establish the main results on periodic structure of $\sigma$-permutation maps on $\mathbb{T}^n$. The converse problem of constructing maps on $\mathbb{T}^n$ with a prescribed set of periods is also solved. As a corollary of the

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main results we give a full characterization of the finite sets of periods which a σ-permutation map can have.

2. Properties of σ-permutation maps

The general notation and terminology used here is the same as in [BL].

The permutation σ = (2 3 ... n 1) is called the standard cyclic permutation of \{1, 2, ..., n\}. Note that Lemma 3.2 in [BL] for \( f^n \) holds also for the space \( X^n \). Therefore we may always assume in the proofs that σ is the standard cyclic permutation of \{1, 2, ..., n\}.

For \( i \in \{1, 2, ..., n\} \) and \( j \in \{0, 1, 2, ...\} \) define \( f_i^{(j)} : X \to X \) by \( f_i^{(0)}(x) = x \), and \( f_i^{(j)}(x) = f_{\sigma(i)}(f_{\sigma^2(i)}(x)) \), if \( j \geq 1 \). This means that for \( 1 \leq i \leq n \) and \( j \geq 1 \) we have

\[
f_i^{(j)} = f_{\sigma(i)} \circ f_{\sigma^2(i)} \circ ... \circ f_{\sigma^j(i)}.
\]

For the standard cyclic permutation we use the convention \( x_\alpha = x_\beta \) if and only if \( \alpha \equiv \beta \pmod{n} \), and the same for \( f_\alpha = f_\beta \). In this case \( f_i^{(j)} = f_{i+1} \circ f_{i+2} \circ ... \circ f_{i+j} \) holds for \( 1 \leq i \leq n \) and \( j \geq 1 \).

Lemmas 3.1, 3.3 and 3.4 of [BL] remain valid in the general setting of metric spaces. In the following result the statements of those lemmas are adapted to this situation.

**Lemma 2.1.** For a σ-permutation map \( F : X^n \to X^n \) the following properties hold:

1. Fix \( k \geq 1 \). Then \( x \) is a fixed point of \( F^k \) if and only if \( x \) is an \( m \)-periodic point of \( F \) with \( m \mid k \). For any \( i \in \{1, 2, ..., n\} \), \( x \) is a fixed point of \( \left(f_i^{(n)}\right)^k \) if and only if \( x \) is an \( m \)-periodic point of \( f_i^{(n)} \) with \( m \mid k \).
2. \( f_i^{(j_1+j_2)}(x) = f_i^{(j_1)}(f_{\sigma^{j_2}(i)}(x)) \) for every \( j_1, j_2 \geq 0 \) and every \( i \in \{1, 2, ..., n\} \).
3. In the case of the standard cyclic permutation we obtain \( F^{kn+j}((x_i)_{i=1}^n) = \left(f_i^{(j)}(f_{i+j}^n(x_i+j))\right)_{i=1}^n \), for every \( k \geq 0 \) and \( 1 \leq j \leq n \).
4. Let \( f, g : X \to X \) be continuous. If \( x \in X \) is a periodic point of order \( k \) for \( g \circ f \), then \( f(x) \) is a periodic point of order \( k \) for \( f \circ g \), and \( \text{Orb}_{f \circ g}(f(x)) = f \left( \text{Orb}_{g \circ f}(x) \right) \).
5. \( \text{Per}(f_i^{(n)}) = \text{Per}(f_2^{(n)}) = ... = \text{Per}(f_n^{(n)}) \).

Proposition 4.1, Lemma 4.5 and Proposition 4.2 of [BL] also hold in general. We obtain the following result which allows us to know when the divisors of \( n \) are included in \( \text{Per}(F) \).