Wilson’s functional equation in dimension 3

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To the memory of my friend, brother and colleague, Yiannis Kyparissis

Summary. We determine the general solution of the functional equation
\[ f(x + y) + f(x - y) = A(y)f(x) \quad (x, y \in G), \]
where \( G \) is a 2-divisible abelian group, \( A \) is a \( 3 \times 3 \) matrix-valued function and \( f \) is a vector-valued function with linearly independent components. Using this result we solve the scalar equation
\[ f(x + y) + f(x - y) = g_1(x)h_1(y) + g_2(x)h_2(y) + g_3(x)h_3(y) \quad (x, y \in G). \]

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1. Introduction

Let \( G \) be an abelian group divisible by 2 (that is, \( 2G = G \)) and \( F \) an algebraically closed commutative field of characteristic zero. Throughout this paper we denote by \( \mathcal{L}, \mathcal{M}, \mathcal{N} \) the sets of all solutions \( f : G \to F \) of the functional equations
\[ f(x + y) = f(x) + f(y), \quad f(x + y) = f(x)f(y) \quad \text{and} \quad f(x + y) + f(x - y) = 2f(x) + 2f(y), \]
respectively. Also, \( F_n \) is the set of all \( n \times n \) matrices over \( F \), \( F^n \) is the set of all \( n \times 1 \) matrices (column vectors) over \( F \), \( F \) is the set of all vector-valued functions with linearly independent components, \( I_n \) is the unit matrix of \( F_n \) and 0 (the zero of \( F \)) is also used for the zeros of \( G, F^n \) and \( F_n \) (for any \( n \)). The transpose of a matrix \( M \) is denoted by \( M' \). A function \( f : G \to F \) is called additive if \( f \in \mathcal{L} \).

Throughout this paper, the letters \( \phi, \chi, \nu \) (along with subscripts) are used exclusively for denoting elements of \( \mathcal{L}, \mathcal{M}, \mathcal{N} \), respectively. If not otherwise specified, the letters \( \gamma, \delta, \kappa, \lambda, \mu \) denote elements of \( F \).

The functional equation
\[ f(x + y) + f(x - y) = A(y)f(x) \quad (x, y \in G), \quad \text{(W)} \]
with \( A : G \to F_n \) and \( f : G \to F^n \), can be considered as an \( n \)-dimensional version
of the scalar Wilson equation \( f(x + y) + f(x - y) = 2f(x)g(y) \) introduced in [22].

For \( n = 2 \) the equation (W) was solved in [14, Theorem 1] and, in a more general setting, in [18, Theorem 3.3]. Here we solve it in the case \( n = 3 \). It turns out that the solutions are of the same pattern as in the 2-dimensional case.

Theorem 2 below gives the main result of this paper. Using this result we solve in Theorem 3 the scalar equation

\[
f(x + y) + f(x - y) = g_1(x)h_1(y) + g_2(x)h_2(y) + g_3(x)h_3(y).
\]

Special cases of this equation have been extensively studied under various conditions by many authors (see, e.g., [1], [2], [3], [5], [6], [7], [14], [17]).

2. Preliminary results

We begin with a result of linear algebra which is a basic tool in what follows.

**Lemma 1.** If \( A : G \to F_3 \) is a function satisfying \( A(x)A(y) = A(y)A(x) \) for all \( x, y \in G \), then it can be written in the form \( A(x) = CB(x)C^{-1} \), where

\[
B(x) = \begin{pmatrix}
b_1(x) & \lambda_1b(x) & b_0(x) \\
0 & b_2(x) & \lambda_2b(x) \\
0 & 0 & b_3(x)
\end{pmatrix},
\]

\( C \in F_3 \) is an invertible matrix, \( \lambda_1, \lambda_2 \in F \) are constants, and the functions \( b, b_0, b_1, b_2, b_3 : G \to F \) fulfil one of the following conditions:

1. \( b(x) \equiv b_0(x) \equiv 0 \),
2. \( b_1(x) \equiv b_2(x) \not\equiv b_3(x) \) and \( b_0(x) \equiv \lambda_2 = 0 \),
3. \( b_1(x) \equiv b_2(x) \equiv b_3(x) \).

**Proof.** By linear algebra (see, e.g., [19], [12]) we know that the matrices \( A(x) \) can be written in the form \( A(x) = CB(x)C^{-1} \), where \( C \in F_3 \),

\[
B(x) = \begin{pmatrix}
b_{11}(x) & b_{12}(x) & b_{13}(x) \\
0 & b_{22}(x) & b_{23}(x) \\
0 & 0 & b_{33}(x)
\end{pmatrix},
\]

and the functions \( b_{ij} \) fulfil one of the following conditions:

1. \( b_{12} = b_{13} = b_{23} = 0 \),
2. \( b_{11} = b_{22} \not\equiv b_{33} \) and \( b_{13} = b_{23} = 0 \),
3. \( b_{11} = b_{22} = b_{33} \),
4. \( b_{11} \not\equiv b_{22} = b_{33} \) and \( b_{12} = b_{13} = 0 \).

The cases (i), (ii) lead to (1) and (2), respectively. Since \( B(x)B(y) = B(y)B(x) \) the case (iii) gives \( b_{12}(x)b_{23}(y) = b_{12}(y)b_{23}(x) \) which means that \( b_{12}, b_{23} \) are linearly dependent and can be written as \( b_{12} = \lambda_1b, b_{23} = \lambda_2b \). So we get (3). Finally