On a functional equation related to power means

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Summary. In the paper On the mutual noncompatibility of homogeneous analytic non-power means (Aequationes Math. 45 (1993)) M. E. Kuczma considered analytic solutions of the functional equation

\[ x + g(y + f(x)) = y + g(x + f(y)) \]

on the real line. Solutions in the class of twice differentiable functions are given in the author’s paper Differentiable solutions of a functional equation related to the non-power means (Aequationes Math. 55 (1998)). During the 38th International Symposium on Functional Equations N. Brillouët-Belluot presented the proof that differentiable solutions of this equation have the same form as in the previous cases.

We present solutions of the equation in the class of monotonic Jensen convex or Jensen concave functions on the real line. This time we get already some new families of solutions.

Mathematics Subject Classification (2000). 39B22.

Keywords. Jensen convex, Jensen concave functions.

1. Introduction

While studying the problem of compatibility of means M. E. Kuczma [5] had to deal with the functional equation

\[ x + g(y + f(x)) = y + g(x + f(y)), \]

where \( f, g \) are unknown functions from the real line \( \mathbb{R} \) into itself. In [5] equation (1) was solved in the class of analytic functions and formal power series. Regardless of the problem of compatibility of means it seems that (1) itself deserves further investigations.

Usually equations of several variables furnish much information on the unknown function(s). However, that is not the case for (1); actually this equation says only that \( f \) and \( g \) are functions admitting the commutativity of the binary operation

\( (x, y) \mapsto x + g(y + f(x)) \).

We are faced with a similar problem while looking for solutions of the fundamental equation of information, for instance (see J. Aczél & Z. Daróczy [1]).
In [6] all twice differentiable solutions of (1) were described. N. Brillouët-Belluot showed that differentiable solutions of this equation have the same form as in the previous cases ([2], [3]). Some other studies of the equation may be found in the author’s paper [7].

In the sequel we always understand by a solution of (1) a pair of real functions \((f, g)\), defined on the whole real line, which satisfies (1) for all \(x, y \in \mathbb{R}\).

Following the earlier papers (see for example [6], Remark, p. 147), it is enough to investigate those solutions \((f, g)\) of (1) which vanish at zero. So, we will consider the system

\[
\begin{align*}
    x + g(y + f(x)) &= y + g(x + f(y)) \\
    f(0) &= g(0) = 0.
\end{align*}
\]

(10)

Let, moreover, throughout the paper, \(\mathbb{R}_+\) stand for the interval \([0, +\infty)\).

Some earlier results concerning \((10)\) are stated in the following theorem.

**Theorem I.** Let \(A\) be the class of analytic functions or the class of twice differentiable functions or the class of differentiable functions. Then the general solution \((f, g)\) of \((10)\) in \(A\) is given by the formulas

\[
\begin{cases}
    f(x) = ax \\
    g(x) = \frac{1}{1-a}x,
\end{cases}
\]

where \(a \in \mathbb{R} \setminus \{-1, 1\}\) is arbitrarily fixed;

\[
\begin{cases}
    f(x) = -x \\
    g(x) = \frac{1}{2}x + p(x),
\end{cases}
\]

where \(p : \mathbb{R} \to \mathbb{R}\) is an arbitrary even function from the class \(A\) such that \(p(0) = 0\);

\[
\begin{cases}
    f(x) = -\frac{1}{\gamma} \ln(ae^{\gamma x} + (1 - a)) \\
    g(x) = -\frac{1}{\gamma} \ln\left(\frac{1}{1+a}e^{-\gamma x} + \frac{a}{1-a}\right),
\end{cases}
\]

where \(\gamma \in \mathbb{R} \setminus \{0\}\) and \(a \in (0, 1)\) are arbitrarily fixed.

Conversely, each of the three pairs of functions listed above yields a solution of \((10)\) in the class \(A\).

The aim of the present paper is to solve \((10)\) in the class of monotone Jensen convex or Jensen concave functions on the real line, so in a special class of continuous functions.

We start with an easy lemma.

**Lemma 1.** Let \((f, g)\) be a solution of \((10)\). If \(f\) is decreasing, then \(g\) is increasing.

**Proof.** Take \(x, y \in \mathbb{R}\) such that \(x < y\). Then from (1) we get \(g(y + f(x)) > g(x + f(y))\). Due to the fact that \(f\) is decreasing, we have \(f(x) \geq f(y)\), and so \(y + f(x) > x + f(y)\). Since \(g\) is monotone, it must be increasing.