On the determination of left-continuous t-norms and continuous archimedean t-norms on some segments*

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Summary. The question of determining uniquely either a continuous Archimedean t-norm or a left-continuous t-norm on some vertical segments and/or on some level sets of its graph is investigated.

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1. Introduction

Associative functions on real intervals were first considered by Abel in 1826 ([1]) and have since been studied by many other mathematicians – see the classic treatise by J. Aczél ([2]) for mathematical and historical details. A special class of associative functions, the so-called t-norms, have been applied in various mathematical disciplines including game theory, the theory of non-additive measures and integrals, the theory of measure-free conditioning, fuzzy set theory, fuzzy logic, fuzzy control, preference modelling and decision analysis, and artificial intelligence since their introduction in 1942. They have been studied not only with their original application to probabilistic metric spaces in [12], but also, in connection with semigroup theory and functional equations. For further details we refer the reader to the monograph on t-norms ([10]).

Many authors have focused on the identification of small subsets of the unit square which uniquely determine a continuous Archimedean t-norm ([9, 12]). The main results of such investigations are the following:

A strict t-norm $T$ is uniquely determined by its diagonal section and the section along the graph of a strictly decreasing bijection of the unit interval. Moreover, in [3, 5] those requirements are weakened considerably. In [3] it was shown that it suffices to know the values of a strict t-norm on some appropriate subset of

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the two diagonals, e.g., on \( \{(x, x) | x \in [0, 1]\} \) and on \( \{(x, 1-x) | x \in [0, \varepsilon]\} \), for any \( \varepsilon \in ]0, 1[ \).

A nilpotent t-norm is uniquely determined by its diagonal section and its preimage of \( \{0\} \) ([4]).

In the present paper other subsets of the unit square are shown to have the property that there exists a unique t-norm (either a nilpotent one or a strict one or a left-continuous one) provided that its values are given on that subset. These subsets are either vertical segments of the graph of the t-norm \( T \), that is, one-place functions of the form \( T(., x) \), which can be considered as intersections of the graph of the t-norm with vertical planes, or horizontal segments, that is, one-place functions of the form \( f_c(x) \) (see Definition 3), which can be considered as border lines of intersections of the graph of the t-norm with horizontal planes.

2. Preliminaries

A triangular norm (t-norm for short) is a function \( T : [0, 1]^2 \to [0, 1] \) such that for all \( x, y, z \in [0, 1] \) the following four axioms (T1)–(T4) are satisfied.

\[
\begin{align*}
\text{(T1) Commutativity: } & \quad T(x, y) = T(y, x) \\
\text{(T2) Associativity: } & \quad T(x, T(y, z)) = T(T(x, y), z) \\
\text{(T3) Monotonicity: } & \quad T(x, y) \leq T(x, z) \quad \text{whenever } y \leq z \\
\text{(T4) Boundary condition: } & \quad T(x, 1) = x \\
\text{(T5) Boundary condition: } & \quad T(0, y) = 0 \\
\text{(T6) Conjunctive nature: } & \quad T(x, y) \leq \min(x, y).
\end{align*}
\]

It can immediately be seen that the first four axioms imply (T5) and (T6). A t-norm is called continuous, if it is continuous as a two-place function. A continuous t-norm is called Archimedean if \( T(x, x) < x \) holds for \( x \in ]0, 1[ \). A continuous Archimedean t-norm is called nilpotent, if it has zero divisors (that is, if there exists \( x \in ]0, 1[ \) such that there exists \( y \in ]0, 1[ \) with \( T(x, y) = 0 \). A prototype of nilpotent t-norms is the so-called Lukasiewicz t-norm, given by

\[ T_L(x, y) = \max(0, x + y - 1). \]

A continuous Archimedean t-norm is called strict if it has no zero divisors. An example is the product t-norm, given by

\[ T_P(x, y) = x \cdot y. \]

In fact, these are the unique examples for nilpotent and for strict t-norms up to \( \varphi \)-transformations, as it is shown by the following theorem.