Trigonometric formulas and $\mu$-spherical functions

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Summary. Our main goal is to determine the continuous and bounded solutions of the functional equations
\[
\int_G f(xt\sigma(y))d\mu(t) = f(x)g(y) + f(y)g(x), \quad x, y \in G,
\]
and
\[
\int_G g(xt\sigma(y))d\mu(t) = g(y)g(x) + f(x)f(y), \quad x, y \in G,
\]
where $G$ is a locally compact group, $\sigma$ is a continuous homomorphism such that $\sigma \circ \sigma = I$ and $\mu$ is a $\sigma$-invariant complex bounded measure on $G$. The solutions are expressed by means of $\mu$-spherical functions and solutions of the functional equation
\[
\int_G f(xty)d\mu(t) = f(x)\varphi(y) + f(y)\varphi(x), \quad x, y \in G,
\]
in which $\varphi$ is a $\mu$-spherical function. The results obtained in the present paper are a natural extension of previous work by Poulsen and Stetkær in which $\mu$ is the Dirac measure concentrated at the identity element of $G$.

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1. Introduction

A number of results has been obtained for the trigonometric addition and subtraction formulas and their generalization, Levi-Civitâ’s equation
\[
f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y), \quad x, y \in G,
\]
where the functions $f, g_i, h_i$ to be determined are complex valued functions on a group $G$. The most comprehensive recent studies of the trigonometric addition and subtraction formulas are due to Poulsen and Stetkær ([7]) who found the complete
set of solutions of the functional equations
\[ f(x\sigma(y)) = f(x)g(y) + f(y)g(x), \quad x, y \in G, \tag{1.2} \]
\[ f(x\sigma(y)) = f(x)g(y) - f(y)g(x), \quad x, y \in G, \tag{1.3} \]
and
\[ f(x\sigma(y)) = f(x)f(y) + g(y)g(x), \quad x, y \in G, \tag{1.4} \]
where
\[ \sigma(xy) = \sigma(x)\sigma(y), \quad \sigma \circ \sigma(x) = x, \quad x, y \in G, \tag{1.5} \]
and by Chung, Kannappan and Ng in [5] who solved the functional equation
\[ f(xy) = f(x)g(y) + f(y)g(x) + h(x)h(y), \quad x, y \in G. \tag{1.6} \]

As a continuation and a generalization of these investigations we determine the continuous and bounded solutions of the following functional equations
\[ \int_G f(x\sigma(y))d\mu(t) = f(x)g(y), \quad x, y \in G, \tag{1.7} \]
\[ \int_G f(x\sigma(y))d\mu(t) = f(x)g(y) - f(y)g(x), \quad x, y \in G, \tag{1.8} \]
and
\[ \int_G g(x\sigma(y))d\mu(t) = g(y)g(x) + f(x)f(y), \quad x, y \in G, \tag{1.9} \]
with the only assumption that \( \mu \) is a \( \sigma \)-invariant complex bounded measure on \( G \).

In the particular case, when \( \mu \) is a compactly supported measure on \( G \), the solutions may be sought in the space of the continuous functions on \( G \).

The paper [7] of Poulsen and Stetkær just mentioned is the essential motivation for the present work and the methods used here are closely related to and inspired by those in [7].

The paper has the following content:

In Sections 2 and 3, we give some definitions and prove some theorems, which will be used in the proof of our results.

In Section 4, we will find the solutions of equation (1.7). The results are presented in Theorem 4.1.

Section 5 is devoted to the functional equation (1.8). The solution formulas are given in Theorem 5.1.

Section 6 includes a complete description of the solutions of equation (1.9).

The solutions of the functional equations considered in this work are expressed by means of \( \mu \)-spherical functions (see Definition 2.1 below) and solutions of the functional equation
\[ \int_G f(x\sigma(y))d\mu(t) = f(x)\varphi(y) + f(y)\varphi(x), \quad x, y \in G, \tag{1.10} \]