Euclidean geometry of orthogonality of subspaces

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Summary. We prove that Euclidean geometry is interpretable in terms of orthogonality of its $k$-subspaces, and thus it can be formalized as a theory of such an orthogonality.

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Introduction

The aim of this note is to bring together some results concerning the adjacency of subspaces of an affine space and orthogonality in a Euclidean space, and to show that Euclidean geometry can be expressed as a theory of orthogonality of subspaces. The result obtained generalizes the old result from the foundations of geometry proved by Schwabhäuser and Szczerba in [20] which states that Euclidean geometry (in dimensions $\geq 4$) can be expressed as the theory of orthogonality of lines.

That result arose from investigations on possible primitive notions for various classical geometries with point individuals replaced by other, simple and natural, geometrical objects (e.g. plane Euclidean geometry as theory of tangency of circles, in [19], Euclidean and hyperbolic geometries as theories of tangency of spheres, see [11]). Among others, intersection of lines was proved to be such a notion for affine and projective geometries (of suitable dimensions, see [5], [12]). In every case a problem appears, which relations on this new universe are sufficient for the geometry in question; in particular: can we find a single sufficient binary relation? In most cases the answer is affirmative (we have quoted tangency of circles and tangency of spheres, line intersection; we should also mention parallelism of planes in hyperbolic geometry, see [13, 17], orthogonality of lines in a hyperbolic plane, confer [14, 9, 10]).

Simultaneously, many results have been obtained concerning adjacency of subspaces of projective spaces, null systems and similar structures $\mathfrak{M}$, starting from
the classical Chow Theorem (cf. [2]), most of them stating that

*a bijection of $k$-subspaces of $\mathcal{M}$ that preserves the adjacency of these subspaces is determined by an automorphism of the underlying space $\mathcal{M}$* \hfill (*)

(cf. e.g. [15], [16], [17], [18]). Usually, two $k$-subspaces are called *adjacent* when they share a common $(k - 1)$-subspace. For a discussion of various possibly more general meanings of the term *adjacency* of subspaces of $\mathcal{M}$ in the framework of the Grassmann space of $\mathcal{M}$ see e.g. [16], [18]. The term adjacency (of matrices) appears also basic in the so called geometry of matrices (cf. origins in [6], the monograph [22], and many papers, e.g. [3, 21]).

In the original fundamental papers on that subject (comp. [2], [4]) theorems like (\#) were formulated and proved directly in the language of linear algebra. However, a more elegant way to obtain such a result consists in proving that

*the notion of a point of $\mathcal{M}$ and after that basic relations of $\mathcal{M}$ are definable in terms of the adjacency of $k$-subspaces of $\mathcal{M}$.* \hfill (\###)

Then (\#) becomes a direct consequence of (\###). Moreover, from (\###) we get immediately that the geometry of $\mathcal{M}$ can be expressed as the theory of the adjacency of its $k$-subspaces.

From this modern point of view the classical result of Schwabhäuser and Szczerba mentioned at the beginning is somehow evident:

- It is known that from the line intersection we can interpret the notion of a point (in the affine geometry of the considered Euclidean space), the point-line incidence, and the parallelism of lines.
- Clearly, given the notion of a point, point-line incidence, and the orthogonality of lines, we have interpreted the underlying Euclidean space.

Thus the problem reduces to define line intersection in terms of orthogonality. In fact, this was the way in which the result was originally proved in [20].

As a first step of the process described above, *lines* can be replaced by $k$-*subspaces* (the dimension of the geometry being $\geq k+2$) and *intersection* by *adjacency*. Thus it suffices to show that adjacency is definable in terms of orthogonality to obtain that this orthogonality can be used as a single primitive notion in Euclidean geometry. This is our goal in this note.

Some additional comments are needed, however: what does the term *orthogonality* mean when applied to subspaces of higher dimensions? Frequently, lines (and, generally, subspaces) are called orthogonal if they have orthogonal directions, where the direction of a subspace is a suitable vector subspace of the metric vector space which represents the considered Euclidean space. This notion, however, is insufficient to express Euclidean geometry (cf. 1.1). Therefore, we must additionally require that orthogonal subspaces intersect; this variant of orthogonality will be called *perpendicularity* of subspaces and will be denoted by $\perp$.