Properties of d’Alembert functions

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Summary. We study properties of solutions $f$ of d’Alembert’s functional equations on a topological group $G$. For nilpotent groups and for connected, solvable Lie groups $G$, we prove that $f$ has the form $f(x) = (\gamma(x) + \gamma(x^{-1}))/2$, $x \in G$, where $\gamma$ is a continuous homomorphism of $G$ into the multiplicative group $\mathbb{C} \setminus \{0\}$. We give conditions on $G$ and/or $f$ for equality in the inclusion $\{u \in G \mid f(xu) = f(x) \text{ for all } x \in G\} \subseteq \{u \in G \mid f(u) = 1\}$.

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1. Introduction and notation

Let $G$ be a topological group. In this paper we will study properties of the solutions $f : G \to \mathbb{C}$ of d’Alembert’s functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \quad \text{for all } x, y \in G. \quad (1.1)$$

A continuous solution $f$ such that $f(e) = 1$, $e$ denoting the neutral element of $G$, is said to be a d’Alembert function on $G$. The condition $f(e) = 1$ may be replaced by the equivalent one that $f \neq 0$.

Our purpose is to investigate the d’Alembert functions on groups that need not be compact or abelian, and to find the properties that such functions possess. The d’Alembert functions are known on abelian groups, where Kannappan [10] characterized them (see Theorem 2.12), and they have also been studied on compact groups by Davison in [4] and Yang in [14].

Any solution of (1.1) is also a solution of d’Alembert’s long functional equation

$$f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y), \quad x, y \in G. \quad (1.2)$$

Some of our results are valid for solutions of (1.2).

Three themes about a function $f \in C(G)$ dominate the paper:

(I) The first and longest is about the following three subsets of $G$ and their
mutual relations:
\[ N(f) := \{ u \in G \mid f(ux) = f(x) \text{ for all } x \in G \}, \quad (1.3) \]
\[ U(f) := \{ u \in G \mid f(u) = f(e) \}, \quad (1.4) \]
\[ Z(f) := \{ z \in G \mid f(xyz) = f(xzy) \text{ for all } x, y \in G \}. \quad (1.5) \]

In particular we ask when \( f \) satisfies Kannappan’s condition \( Z(f) = G \). In other words, when \( f(xyz) = f(xzy) \) for all \( x, y, z \in G \). Following Davison [4] we say that a d’Alembert function is abelian if it satisfies Kannappan’s condition, and non-abelian if it does not.

(II) The second is the range of a possibly unbounded solution of d’Alembert’s long functional equation.

(III) The third is about properties that d’Alembert functions have in common with group characters, i.e. properties from abstract harmonic analysis.

Some comments on the themes:

(I) \( N(f) \) is the group of (right) periods of the function \( f \). For the cosine function the set of periods \( \{ u \in \mathbb{R} \mid \cos(x + u) = \cos x \text{ for all } x \in \mathbb{R} \} \) and the set \( \{ u \in \mathbb{R} \mid \cos u = 1 \} \) are the same. This equality persists for d’Alembert functions on nilpotent (and hence also on abelian) groups and on connected, solvable Lie groups, although not on all groups (Proposition 4.1, Theorem 5.2, Remark 2.2).

In [4] Davison calls \( N(f) \) the nub of \( f \). In hindsight the nub \( N(f) \) of a d’Alembert function was in disguise introduced already in the paper [12] (the group \( H_1 \) in [12, Proposition 6.3]), but it was not exploited beyond the study of d’Alembert’s functions on step 2 nilpotent groups in [12]. Its role was first recognized by [4]. The discussion in the present paper of its relation to \( U(f) \) is new.

D’Alembert functions need not be abelian (Subsections 8.2–8.4 provide examples), but we prove that they are so on groups that are close to being abelian (Theorem 4.2 and Theorem 5.2). Furthermore we find and apply a new criterion for a d’Alembert function \( f \) to be abelian: \( f \) is abelian iff \( f([x, y]) = 1 \) for all \( x, y \in G \) (Theorem 3.7(b)).

In the paper [10] from 1968 Kannappan proved that if a d’Alembert function \( f \) is abelian (in particular if \( G \) is abelian) then it can be written in the form \( f = (\gamma + \gamma)/2 \) where \( \gamma : G \to \mathbb{C}^* \) is a continuous homomorphism. Before that d’Alembert functions were only known on special groups like the real line. Sufficient conditions for d’Alembert functions on non-abelian groups to be abelian have later been published by Corovei [2, 3], Friis [7] and the author [11, 12]. Most of these results are generalized here in Sections 4 and 5.

(II) In Section 6 we study solutions of d’Alembert’s long functional equation, that may be unbounded. Davison ([4]) considered the range of bounded d’Alembert functions.

(III) In Section 7 we present two properties from abstract harmonic analysis: Linear independence and orthogonality relations.