The projective translation equation and rational plane flows. I

Giedrius Alkauskas

Abstract. Let \( x = (x, y) \). A plane flow is a function \( F(x, t) : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2 \) such that \( F(F(x, s), t) = F(x, s + t) \) for (almost) all real numbers \( x, y, s, t \) (the function \( F \) might not be well-defined for certain \( x, t \)). In this paper we investigate rational plane flows which are of the form \( F(x, t) = \phi(xt)/t \); here \( \phi \) is a pair of rational functions in 2 real variables. These may be called projective flows, and for a description of such flows only the knowledge of Cremona group in dimension 1 is needed. Thus, the aim of this work is to completely describe over \( \mathbb{R} \) all rational solutions of the two dimensional translation equation \((1 - z)\phi(x) = \phi(\phi(xz)(1 - z)/z)\). We show that, up to conjugation with a 1-homogenic birational plane transformation (1-BIR), all solutions are as follows: a zero flow, two singular flows, and one non-singular flow for each non-negative integer \( N \), called the level of the flow. The case \( N = 0 \) stands apart, while the case \( N = 1 \) has special features as well. Conjugation of these canonical solutions with 1-BIR produce a variety of flows with different properties and invariants, depending on the level and on the conjugation itself. We explore many more features of these flows; for example, there are 1, 4, and 2 essentially different symmetric flows in cases \( N = 0, N = 1 \), and \( N \geq 2 \), respectively. Many more questions related to rational flows will be treated in the second part of this work.

Mathematics Subject Classification (2000). Primary 39B12, 14E07; Secondary 35F05, 37E35.

Keywords. Translation equation, flow, projective geometry, rational functions, rational vector fields, iterable functions, birational transformations, involutions, Cremona group, linear ODE, linear PDE.

Contents

Terminology 274

1. Introduction and main results 275

1.1. The affine translation equation 276
1.2. The projective translation equation 277
1.3. Singular and non-singular solutions 278
1.4. Main results 279
1.5. Flows in univariate form 281

2. Basic examples and properties of flows 283
2.1. The degenerate case 283
2.2. Level 0 flows 285
2.3. Level 1 flows 286
2.4. Level $N \geq 2$ flows 288

3. Vector field of a rational flow and PDEs 292
3.1. Basic properties of a vector field 292
3.2. Partial differential equations 295

4. The proof of the Theorem 298
4.1. Step I 299
4.2. Step II 306
4.3. Step III 308
4.4. Step IV 313
4.5. The maps $p$ and $\hat{p}$ 315
4.6. Reconstruction of the level from the vector field 317

5. More properties of flows; final remarks 318
5.1. The orbits of a point and the level of a flow 318
5.2. Pseudo-flows of level 0 319
5.3. Symmetric flows 320
5.4. Duality map in $\text{FL}_N$ 323
5.5. Surfaces, invariant measures, dynamics 324
5.6. Higher dimensional and other investigations 324

Appendix A. Summary of facts from birational geometry 325
A.1. Birational 1-homogenic maps 326
A.2. Involution 327

References 328

Terminology

Many of the classical terms from geometry are new in the field of translation equation, so we briefly recall the definitions. Also, we introduce some new terms.

**1-BIR:** A birational transformation of affine space such that it is given by a collection of 1-homogenic rational functions.

**Birational transformation:** A map from the affine or projective space to itself which is rational, and whose inverse is also rational.

**Affine translation equation:** the functional equation (1).