Invariance under outer inverses

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Abstract. We shall use the minus partial order combined with Pierce’s decomposition to derive the class of outer inverses for idempotents, units and group invertible elements. Subsequently we show, for matrices over a field $F$, that the triplet $B\hat{A}C$ is invariant under all choices of outer inverses of $A$ if and only if $B = 0$ or $C = 0$.

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1. Introduction

Let $R$ be a ring with 1.

An element $a$ is called regular if $aa^-a = a$ for some inner or 1-inverse $a^-$. The condition for regularity is a linear condition, and the set of all inner inverses is given by

$$\{a^{(1)}\} = a^- + (1 - a^- a)R + R(1 - aa^-).$$

An outer or 2-inverse $\hat{a}$ of an element $a$ is such that $\hat{a}a\hat{a} = \hat{a}$. It is a quadratic condition in $\hat{a}$. It is clear that $\hat{a}a\hat{a}$ will always be regular.

A 1–2 or reflexive inverse of $a$ is denoted by $a^+$ and satisfies

$$aa^+a = a$$

and $a^+aa^+ = a^+$. The set of all outer inverses of an element $a$ will be denoted by $T_a$ or $\{a^{(2)}\}$ and the set of all idempotents will be denoted by $E$. It is clear that a regular element $a$ admits $a^-aa^-$ as an outer inverse.

Given the quadratic nature of the outer inverse condition, the characterization of $T_a$ remains clouded in general. We shall characterize $T_a$ for several

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special types of elements, such as idempotents and units. We then use these results to nail down \( T_a \) for group invertible elements.

We shall also use the concepts of a unit-regular element \( a \), for which there is a unit \( u \) in \( R \) such that \( aua = a \), and (ii) that of a prime ring, for which \( aRb = (0) \), forces either \( a = 0 \) or \( b = 0 \). It should be noted that \( a \) is unit regular if and only if \( a = ppeq \), for some unit \( p \) and \( q \) and some idempotent \( e \).

2. Classes of outer inverses

We begin by recalling [7] that for any \( p, a \) and \( q \), and any \( \overline{(qap)} \)

\[
p(\overline{qap})q \cdot a \cdot p(\overline{qap})q = p(\overline{qap})q
\]

so that it is prudent to define, for a fixed \( p \) and \( q \),

**Definition 2.1.** \( S_{p,q} = \{ p(\overline{qap})q; \text{any } \overline{(qap)} \} \)

Clearly

\[
S_{p,q} = p \cdot T_{qap} \cdot q \subseteq T_a = S_{1,1}.
\]

We next turn to the set of all outer inverses of 1. It precisely equals is the set of all idempotents \( E \), since \( x \cdot 1 \cdot x = x \) if and only if \( x \) is idempotent.

Next we recall that for two units \( p \) and \( q \)

\[
T_{paq} = q^{-1}T_a p^{-1}
\]

To characterize \( T_e \) where \( e \) is idempotent, we make use of the minus order as defined in [3] for a regular element \( a \)

\[
a \leq b \text{ iff } a^{-}a = a^{-}b \text{ and } aa^{-} = ba^{-}, \text{ for some inner inverse } a^{-}.
\]

The key fact that we need is that given \( e^2 = e \) and \( g \) regular such that \( g \leq e \) then \( g^2 = g = ge = eg \). This also tells us that

**Lemma 2.1.** The following are equivalent, for \( e^2 = e \) and \( g \) regular:

(i) \( g \leq e \).
(ii) \( g = ge = eg = g^2 \).
(iii) \( g = eee \) for some outer inverse \( e \).
(iv) \( g = exe = g^2 \) for some \( x \).

We use this in

**Theorem 2.1.** If \( e \) is idempotent then \( T_e = v(x)gw(y) \), where \( v(x) = 1 + (1 - e)xe \), \( w(y) = 1 + ey(1 - e) \), \( g \leq e \), and \( x \) and \( y \) are arbitrary.

**Proof.** Observe that \( v(x) \) and \( w(y) \) are units for all \( x \) and \( y \) and that \( ev(x) = e = w(y)e \). As such \( v(x)gw(y) \cdot e \cdot v(x)gw(y) = vgegw = vgw \). On the other