Solutions of Abel’s equation in relation to the asymptotic behaviour of linear differential equations

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Summary. The general form of all solutions of a given Abel equation is derived and their stability properties are considered. Solutions with positive continuous derivatives have comparable rates of increase. Applications to the asymptotic behaviour of solutions of oscillatory linear differential equations are also mentioned.


Keywords. Abel equation, stability, linear differential equations, asymptotic properties, distribution of zeros.

1. Definitions, notation and some basic facts

Consider Abel’s equation

$$\alpha(\varphi(t)) = \alpha(t) + 1,$$  \hspace{1cm} (A\varphi)

where \( \varphi \) is a continuous strictly increasing real-valued function defined on a half-open interval \([a, b) \subseteq \mathbb{R}, b \leq \infty \), mapping it on a half-open interval \([c, b) \), i.e. \( \varphi \in C^0[a, b), \varphi[a, b) = [c, b) \). Moreover, let \( \varphi(t) > t \) for all \( t \in [a, b) \). These conditions on \( \varphi \) will be always supposed throughout this paper. Let \( \varphi^n \) denote the \( n \)-th iterate of \( \varphi \), i.e. \( \varphi = \varphi^1, \varphi^{n+1} = \varphi \circ \varphi^n \). Evidently \( \varphi^n \) exists for every \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \varphi^n(t) = b \) for any \( t \in [a, b) \). The following results are due to Choczewski [3] and Barvínek [1], see also Kuczma [4].

Remark 1. Under the above conditions, there always exists a solution \( \alpha \) of \((A\varphi)\). There is a unique solution with prescribed values on the interval \([a, \varphi(a)) \). If, moreover, it is continuous on \([a, \varphi(a)) \) and

$$\lim_{t \to \varphi(a)_-} \alpha(t) = \alpha(a) + 1,$$

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then $\alpha$ is continuous on $[a, b]$.

Let $\varphi$ be $n$-times continuously differentiable on $[a, b]$ for some $n \geq 1$. If $\alpha(t)$ is chosen on $[a, \varphi(a))$ so that it is here $n$-times continuously differentiable and

$$\lim_{t \to a^-} (\alpha(\varphi(t)))^{(k)} = (\alpha(t) + 1)^{(k)} \bigg|_{t=a}, \quad k = 0, 1, \ldots, n,$$

then the solution $\alpha$ is also $n$-times continuously differentiable on the whole $[a, b)$, i.e. $\alpha \in C^n[a, b)$. If, moreover, $\alpha'(t)$ is positive on $[a, \varphi(a))$ and $\varphi'(t)$ is positive on $[a, b)$, then also $\alpha'(t) > 0$ on $[a, b)$.

2. Main results

**Theorem 1.** Let $\alpha$ be a continuous and strictly increasing solution of $(A\varphi)$. Then the general solution $\beta$ of Abel’s equation $(A\varphi)$ is

$$\beta(t) = \alpha(t) + P(\alpha(t)),$$

where $P$ is a periodic function with period 1, defined on $[\alpha(a), \infty)$.

**Proof.** Evidently, the function $\alpha$ is defined on $[a, b)$ and maps this interval onto the interval $[\alpha(a), \infty)$. Let $\alpha^{-1}$ be its inverse.

Let $\beta$ be a solution of $(A\varphi)$. Define

$$P(x) := \beta \circ \alpha^{-1}(x) - x, \quad x \in [\alpha(a), \infty).$$

(2)

Now, for $x = \alpha(t)$ we have $P(\alpha(t)) = \beta(t) - \alpha(t)$ and

$$P(x + 1) = P(\alpha(t) + 1) = P(\alpha(\varphi(t))) = \beta(\varphi(t)) - \alpha(\varphi(t))$$

$$= (\beta(t) + 1) - (\alpha(t) + 1) = P(\alpha(t)) = P(x)$$

on $[\alpha(a), \infty)$.

Conversely, let $P$ be a periodic function with the period 1 on $[\alpha(a), \infty)$, and the function $\beta$ be defined as

$$\beta(t) := \alpha(t) + P(\alpha(t)), \quad t \in [a, b).$$

Then $\beta(\varphi(t)) = \alpha(\varphi(t)) + P(\alpha(\varphi(t))) = \alpha(t) + 1 + P(\alpha(t) + 1) = \alpha(t) + 1 + P(\alpha(t)) = \beta(t) + 1$. Thus $\beta$ is a solution of $(A\varphi)$. □

**Remark 2.** If the solution $\beta$ in Theorem 1 is continuous then also the function $P$ in (2) is continuous, and conversely, the continuity of $P$ implies the continuity of $\beta$. 