Some algebraic characterizations of $F$-frames

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In memory of Paul F. Conrad

Abstract. In pointfree topology, $F$-frames have been defined by Ball and Walters-Wayland by means of a frame-theoretic translation of the topological characterization of $F$-spaces as those whose cozero-sets are $C^*$-embedded. This is a departure from the way in which $F$-spaces were defined by Gillman and Henriksen as those spaces $X$ for which the ring $C(X)$ is Bézout, meaning that every finitely generated ideal is principal. In this note, we show that, as in the case of spaces, a frame $L$ is an $F$-frame precisely when the ring $\mathcal{RL}$ of continuous real-valued functions on $L$ is Bézout. A commutative ring with identity is called almost weak Baer if the annihilator of each element is generated by idempotents. We establish that $\mathcal{RL}$ is almost weak Baer iff $L$ is a strongly zero-dimensional $F$-frame.

1. Introduction

In his survey of rings of continuous functions [17], Henriksen laments the fact that certain types of topological spaces that were originally defined in terms of “nice” algebraic properties have come to be known to most topologists by their topological characterizations which, sadly, are considered to be pathological. A case in point is that of $F$-spaces. For these spaces, a familiar topological characterization is that every cozero set is $C^*$-embedded. Their original algebraic definition is that they are spaces $X$ for which $C(X)$ is a Bézout ring, meaning that every finitely generated ideal is principal.

Even in pointfree topology, $F$-frames (which generalize and transcend $F$-spaces) were defined by Ball and Walters-Wayland [1] by a frame-theoretic condition which is the analogue of the topological property already stated, rather than an analogue of the algebraic definition of $F$-spaces. Although several characterizations of $F$-frames $L$ in terms of the ring $\mathcal{RL}$ of continuous real-valued functions on $L$ were obtained in [11], it was not established there that $L$ is an $F$-frame precisely when $\mathcal{RL}$ is a Bézout ring.

In this note we prove this (Proposition 3.2) in hopes it will somewhat ameliorate Henriksen’s lament. One of many algebraic characterizations of $F$-spaces is that $X$ is an $F$-space if and only if the prime ideals of $C(X)$ contained in
any maximal ideal form a chain. In Corollary 3.8, not only do we extend this result to frames, we also add another characterization in terms of $z$-ideals. A key component in proving this is the fact that prime ideals of $RL$ that contain a given one form a chain (Proposition 3.7). This, in turn, is obtained via a result communicated to the author by B. Banaschewski to the effect that radical ideals of $RL$ are absolutely convex (Lemma 3.5). We wind up by giving a characterization in terms of what we call almost round quotient maps. Namely, $L$ is an $F$-frame if and only if every quotient map $\beta L \to M$ is almost round (Proposition 3.11), where, as usual, $\beta L$ denotes the Stone-Čech compactification of $L$.

In the last section, we look briefly at $F'$-frames with the view to augmenting certain results about these types of frames that were obtained in [13]. In particular, it follows from [13, Lemma 4.4] that $\downarrow c$ is an $F'$-frame for each $c \in \text{Coz } L$ whenever $L$ is an $F'$-frame. Here, we strengthen this by showing that the restriction to cozero elements is not necessary (Proposition 4.1). A corollary to this is the fact that a frame is an $F'$-frame if and only if it is locally $F'$ (Corollary 4.2), where the latter is defined as expected.

In [11], we showed that each annihilator of $RL$ is generated by an idempotent if and only if $L$ is extremally disconnected, and that each element-annihilator is generated by an idempotent if and only if $L$ is basically disconnected. Concerning annihilators and idempotents, this begs the questions:

(a) When is every element-annihilator generated by idempotents?
(b) When is every annihilator generated by idempotents?

The answer to the first question, answered in Proposition 4.9, is as stated in the last sentence in the abstract. We give a frame-theoretic characterization of frames with the property mentioned in the second question (Proposition 4.11).

A reader who is familiar with quasi $F$-spaces may wonder why their generalization to frames is not included. Algebraic characterizations of quasi $F$-frames will be presented elsewhere. It should be pointed out that quasi $F$-frames are mentioned in [1] and [19].

2. Preliminaries

For a general theory of frames, we refer to [18]. Here we collect a few facts that will be relevant for our discussion. A frame is a complete lattice $L$ in which the distributive law

$$a \land \bigvee S = \bigvee \{a \land x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of $L$ by $1_L$ and $0_L$, respectively, dropping the decorations if $L$ is clear from the context. The frame of open subsets of a topological space $X$ is denoted by $\mathcal{O}X$. 