A note on representation of lattices by weak congruences

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Abstract. A weak congruence is a symmetric, transitive, and compatible relation. An element \( u \) of an algebraic lattice \( L \) is \( \Delta \)-suitable if there is an isomorphism \( \kappa \) from \( L \) to the lattice of weak congruences of an algebra such that \( \kappa(u) \) is the diagonal relation. Some conditions implying the \( \Delta \)-suitability of \( u \) are presented.

Introduction. Given an algebra \( A = (A, F) \), symmetric and transitive compatible relations on \( A \) are weak congruences. They form an algebraic lattice \( \text{Con}_w A = (\text{Con}_w A, \subseteq) \). For a subuniverse \( B \) of \( A \) (that is, a subalgebra or, absent nullary operations, the empty subset), let \( \Delta_B = \{(x, x) : x \in B \} \). The diagonal \( \Delta = \Delta_A \) plays an important role, since the principal ideal \( \downarrow \Delta \) is isomorphic to the lattice \( \text{Sub} A \) of subalgebras of \( A \), while the principal filter \( \uparrow \Delta \) coincides with the congruence lattice \( \text{Con} A \) of \( A \).

The lattice \( \text{Con}_w A \) of weak congruences of \( A \) was studied in the monograph by B. Šešelja and A. Tepavčević [11] and in many papers including [4, 3, 10, 12, 13, 14, 15, 16, 17]. An element \( a \) of an algebraic lattice \( L \) is \( \Delta \)-suitable if there exists an algebra \( A \) and a lattice isomorphism \( \kappa : L \to \text{Con}_w A \) such that \( \kappa(a) = \Delta \). It is a trivial consequence of the Grätzer-Schmidt Theorem, see [9] or [7], that the bottom element of every algebraic lattice is \( \Delta \)-suitable. So the “real” representation problem, see [17], asks what the \( \Delta \)-suitable elements of algebraic lattices are. This problem has been open since 1989, and we are far from its complete solution. In this paper, our goal is to show that some specific elements are \( \Delta \)-suitable and some construction yields \( \Delta \)-suitable elements.

Basic concepts and notation. An element \( a \) of a complete lattice \( L \) is infinite codistributive if \( a \land \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \land x_i) \) holds for all subsets \( \{x_i : i \in I\} \) of \( L \), see Grätzer and Schmidt [8] or [7]. \( \Delta \)-suitable elements are necessarily infinite codistributive, see [11, Cor. 1.23]. However an infinite codistributive element is not always \( \Delta \)-suitable (see [13] or think of the top element).

For an infinite codistributive element \( a \in L \), the map \( \eta_a : L \to \downarrow a \), where \( x \mapsto a \land x \), is clearly a surjective complete lattice homomorphism. With respect to \( \eta_a \), each \( b \in \downarrow a \) has a greatest preimage which is denoted by \( \mu_a(b) \). Let \( S_a(L) = \{\mu_a(b) : b \in \downarrow a\} \). Notice that \( \mu_a : \downarrow a \to S_a(L) \), where \( b \mapsto \mu_a(b) \), is a...
lattice isomorphism, and its inverse is the restriction \( \eta_a \mid_{S_a(L)} \) of \( \eta_a \) to \( S_a(L) \). Since the kernel of \( \eta_a \) is a complete congruence, it is routine to verify that \( S_a(L) \) is a complete meet-subsemilattice of \( L \). In particular, \( 1 = \bigwedge \emptyset \in S_a(L) \).

If \( a \in L \) and \( (a \lor x) \land (a \lor y) \land (x \lor y) = (a \land x) \lor (a \land y) \lor (x \land y) \) holds for all \( x, y \in L \), then \( a \) is neutral. The center of a bounded lattice is the set of all neutral elements that have complements. We know from Grätzer and Schmidt [8] or [7] that \( a \in L \) belongs to the center of \( L \) iff

\[
L \rightarrow \downarrow a \times \uparrow a, \text{ where } x \mapsto (x \land a, x \lor a), \tag{1}
\]
is a lattice isomorphism.

Next, for a definition, assume that \( K_1 \) and \( K_2 \) are complete lattices, \( E_1 \) is a complete meet-subsemilattice of \( K_1 \), \( E_2 \) is a complete join-subsemilattice of \( K_2 \), and \( \kappa : E_1 \rightarrow E_2 \) is an order isomorphism. (Notice that \( E_1 \) and \( E_2 \) are complete lattices, but \( E_i \) is not a sublattice of \( K_i \) in general.) For \( x \in K_1 \), let \( x^* = \bigwedge (E_1 \cap \uparrow x) \), the "closure of \( x \) with respect to \( E_1 \)". Dually, for \( y \in K_2 \), let \( y_* = \bigvee (E_2 \cap \downarrow y) \). We know from [2, Lemma 2] that for any \( (x, y) \in K_1 \times K_2 \),

\[
\kappa(x^*) \leq y \iff x \leq \kappa^{-1}(y_*) \iff (\exists e \in E_1)(x \leq e \text{ and } \kappa(e) \leq y). \tag{2}
\]

This allows us to define a relation on the (disjoint) union \( K := K_1 \cup K_2 \) in a natural way as follows. For \( x, y \in K \), let \( x \leq y \) iff either \( x, y \in K_i \) and \( x \leq_{K_i} y \) for some \( i \in \{1, 2\} \), or \( x \in K_1 \), \( y \in K_2 \), and any of the (equivalent) conditions in (2) holds. This way, by [1, Proposition 1], \( K \) becomes a complete lattice. Notice that \( K \) is the Graczyńska sum of \( K_1 \) and \( K_2 \), see [5] and [6]; see also [2] for historical remarks. Note that \( K_1 \) is an ideal and \( K_2 \) is a filter of \( K \).

It is straightforward to verify the following statement.

**Lemma 1.** If \( K_1 \) and \( K_2 \) are both algebraic lattices, then \( K \) constructed around (2) is algebraic.

**The results.** While \( \Delta \) is typically distinct from the bottom of \( \text{Con}_w A \), not many general results on \( \Delta \)-suitability of non-zero lattice elements were previously known. Below, we present some results of this kind.

For a \( \Delta \)-suitable element \( a \in L \), we know that \( a \) is infinite codistributive. Therefore, with \( L, S_a(L), \downarrow a, \uparrow a, \) and \( \mu_a^{-1} = \eta_a \mid S_a(L) \) playing the role of \( K_1, E_1, K_2, E_2, \) and \( \kappa \), respectively, we can form the Graczyńska sum described previously. We call this sum, denoted by \( L^{(1a)} \), the \( \downarrow a \)-extension of \( L \).

**Theorem 2.** If \( a \) is a \( \Delta \)-suitable element of an algebraic lattice \( L \) and \( M \) is an algebraic lattice, then \( (a, 1_M) \) is a \( \Delta \)-suitable element of \( L^{(1a)} \times M \).

**Corollary 3.** Let \( a \) belong to the center of an algebraic lattice \( K \). Assume that \( \uparrow a \) has a coatom \( b \) such that \( \uparrow a \setminus \downarrow b = \{1_K\} \). Then \( a \) is a \( \Delta \)-suitable element of \( K \).

These statements allow us to find nonzero \( \Delta \)-suitable elements in direct products of (possibly infinitely many) nontrivial algebraic chains with coatoms. (Note that these direct products are algebraic, and an algebraic chain has a