Boolean-like algebras

ANTONINO SALIBRA, ANTONIO LEDDA, FRANCESCO PAOLI, AND TOMASZ KOWALSKI

Abstract. Using Vaggione’s concept of central element in a double-pointed algebra, we introduce the notion of Boolean-like variety as a generalisation of Boolean algebras to an arbitrary similarity type. Appropriately relaxing the requirement that every element be central in any member of the variety, we obtain the more general class of semi-Boolean-like varieties, which still retain many of the pleasing properties of Boolean algebras. We prove that a double-pointed variety is discriminator if and only if it is semi-Boolean-like, idempotent, and 0-regular. This theorem yields a new Maltsev-style characterisation of double-pointed discriminator varieties.

1. Introduction

Boolean algebras have an exceptionally rich and smooth structure theory, of which Stone’s representation theorem is a prominent example. What is so special about Boolean algebras that is responsible for this nice behaviour? Given a similarity type \( \nu \), can we always find a class of algebras of type \( \nu \) that displays Boolean-like features? And what does it mean, for an algebra of a given type \( \nu \) that may not exhibit such desirable properties, to have at least a subset of Boolean elements that behave well? To address these questions, we use the concept, due to Vaggione [39], of a central element in a double-pointed algebra, meaning an element that induces therein, in a specified sense, a pair of complementary factor congruences. Roughly speaking, given a similarity type \( \nu \) including at least two constants but otherwise fully arbitrary, we associate the presence of a “well-behaved Boolean core” in a \( \nu \)-algebra with the presence of a retract of central elements, and we identify Boolean \( \nu \)-algebras with \( \nu \)-algebras where every element is central. In order to fully appreciate what properties of Boolean algebras are responsible for the most important results concerning this variety, however, the issue is best addressed in a step-by-step fashion. Therefore, following [31], we will decompose the property of centrality into several equational properties, trying to investigate what happens when some of them are satisfied but other ones are dropped. This approach will give rise to a few successive approximations to a full-fledged notion of “Boolean algebra of arbitrary similarity type”.

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Our work ties in nicely with at least three research streams that have received considerable attention in universal algebra and in the investigation into the mathematical foundations of computer science:

- **(Weak) Boolean product representations.** It has been known for a long time that Stone’s representation theorem, perhaps the most distinctive result characterising Boolean algebras (or Boolean rings), can be generalised to a much larger class of algebras. The appropriate tool to attain this goal is the technique of Boolean products, which can be loosened to the notion of weak Boolean product to take care of somewhat less manageable cases (see e.g. [24]). Pierce [36] proved that every commutative ring with unit is representable as a weak Boolean product of directly indecomposable rings; Stone’s representation theorem follows as a corollary by observing that the 2-element ring of truth values is the unique directly indecomposable Boolean ring. The technique of Boolean products underwent remarkable developments over the subsequent years (see e.g. [13, Ch. 4.8]), giving rise to further generalisations of Stone’s theorem by Comer (covering the case of algebras with Boolean factor congruences [16]) and Vaggione [39].

- **Discriminator varieties and noncommutative lattice theory.** Discriminator varieties [41] are referred to by Burris and Sankappanavar as the most successful generalization of Boolean algebras to date, successful because we obtain Boolean product representations (which can be used to provide a deep insight into algebraic and logical properties) [13, p. 186].

  One of the most elegant characterisations of discriminator varieties in the pointed case was obtained by Bignall and Leech [8], who linked them to a noncommutative generalisation of Boolean algebras called left handed skew Boolean ∩-algebras. More precisely, Bignall and Leech proved that: (i) the variety of type (3, 0) generated by the class of all pointed discriminator algebras A = (A; t, 0), where t is the discriminator function on A and 0 is a constant, is term equivalent to the variety of left handed skew Boolean ∩-algebras; (ii) every pointed discriminator variety is term equivalent to a variety of left handed skew Boolean ∩-algebras with additional compatible operations. This result can be easily adapted to the double-pointed case, which is particularly significant in that the variety of Boolean algebras is double-pointed. Following [10], we say that a class of similar algebras is double-pointed if its type has at least two constants that realise distinct elements in any nontrivial member of the class. Some more steps in this direction are taken in what follows.

- **Algebraic investigation of the if-then-else construct.** There is a thriving literature on abstract treatments of the fundamental if-then-else construct of computer science, starting with McCarthy’s seminal investigations [34]. On the algebraic side, one of the most influential approaches originated with a paper by Bergman [6]. Bergman modelled the if-then-else by considering Boolean algebras acting on sets: if the Boolean algebra of actions is the 2-element