Retraction closure property

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Abstract. We say that an algebra $A$ has the retraction closure property (RCP) if the set of all retractions of $A$ is closed with respect to fundamental operations of $A$ applied pointwise. In this paper we investigate this property, both “locally” (one algebra) and “globally” (in some variety of algebras), especially emphasizing the case of groupoids. We compare the retraction closure property with the endomorphism closure property on both levels and prove that a necessary and sufficient condition for a variety $V$ of algebras to have RCP is that $V$ is a variety of entropic algebras that satisfy the diagonal identities.

1. Introduction

The motivation for studying the induced algebraic structure on the set of all retractions of an algebra has its roots in group theory. Namely, let $G = \langle G, + \rangle$ be an Abelian group and $\varphi, \psi$ two endomorphisms of $G$. It is well known that the mapping $\varphi + \psi$, defined pointwise by $(\varphi + \psi)(x) := \varphi(x) + \psi(x)$ for all $x \in G$, is also an endomorphism of $G$. If $G = \langle G, \cdot \rangle$ is a group or a groupoid, the set $\text{End}(G)$ need not be closed with respect to the induced operation. In [3] it is proved that the “sum” of two endomorphisms is an endomorphism in every groupoid in a variety if and only if the variety satisfies the identity $xy \cdot zv \approx xz \cdot yv$, and a variety of groups or loops having the endomorphism closure property consists entirely of Abelian groups. This result has been generalized to arbitrary types of algebras (see [4, 5]).

Another motivation for extending the structure from its base set to the set of all retractions comes from lattice theory. If $L = \langle L, \leq \rangle$ is a lattice, a mapping $\varphi: L \to L$ is a retraction of $L$ if $\varphi$ is an idempotent monotone mapping of the poset $\langle L, \leq \rangle$ into itself. We can turn the set $\text{Retr}(L)$ into a poset by letting $\varphi \leq \psi$ if $(\forall x \in L)\varphi(x) \leq \psi(x)$. Answering the questions posed in [2], it is shown in [6, 7] that the set of all retractions of a lattice is not necessarily a lattice and that $L$ is a complete lattice if and only if the set of retractions is a complete lattice.

Presented by R. Poeschel.
Received April 7, 2012; accepted in final form September 13, 2012.
2010 Mathematics Subject Classification: Primary: 03C05; Secondary: 08A35, 08B99, 20L05.

Key words and phrases: endomorphism, retraction, entropic, medial, diagonal identity, rectangular algebras.

Research supported by the Ministry of Education and Science, Republic of Serbia, Grant No. 174 018.
The notion of the \textit{retraction closure property} was introduced in [8] under a different name ("to be an R-algebra"). We say that an algebra $A$ has the \textit{retraction closure property} if the set $\text{Retr}(A)$ is closed with respect to all fundamental operations of $A$ applied pointwise. In [8], some necessary and some sufficient conditions are given for an algebra to have the retraction closure property. It is shown that this property carries over to retracts of algebras and that almost no classical algebra has the retraction closure property.

The main result of the present paper is that a variety of algebras has the retraction closure property if and only if $V$ is a variety of entropic algebras that satisfy the diagonal identities. The structure of the paper is the following. In Section 2, we introduce the notions of the endomorphism closure property, entropic algebras, the diagonal identity, and the retraction closure property. In Section 3, we study the relationship between the endomorphism closure property (ECP) and the retraction closure property (RCP) for algebras and varieties of algebras. Locally, these two properties are independent: there exist algebras that have ECP and do not have RCP, and conversely. Globally, as a property of a variety, ECP does not imply RCP, but the converse is true: every variety of algebras having RCP also possess ECP. The main result (the characterization of RCP varieties of algebras) is proved in Section 3.

2. Basic definitions

An algebra $A$ is a pair $\langle A, F \rangle$, where $A$ is a non-empty set and $F$ a family of finitary operations on $A$ (called the \textit{fundamental operations} of $A$). The set of all $n$-ary fundamental operations of $A$ is denoted by $F_n$. We will accept the convention to use the same symbol for the functional symbol $f$ of some first order language and its interpretation in an algebra $A$. The set of all endomorphisms of $A$ will be denoted by $\text{End}(A)$. By $\text{Retr}(A)$, we denote the set of all rejections of $A$. For other basic notions concerning universal algebras, we refer to [1].

\textbf{Definition 2.1.} Let $A = \langle A, F \rangle$ be an algebra. For any $f \in F_n$ and any $\varphi_1, \ldots, \varphi_n \in \text{End}(A)$, the \textit{induced mapping} $f^*(\varphi_1, \ldots, \varphi_n) : A \to A$ is defined pointwise, i.e., $f^*(\varphi_1, \ldots, \varphi_n)(x) := f(\varphi_1(x), \ldots, \varphi_n(x))$ for any $x \in A$.

If there is no danger of confusion, instead of $f^*$, we will use the symbol $f$ for this induced mapping too.

\textbf{Definition 2.2.} Let $A = \langle A, F \rangle$ be a universal algebra. We say that $A$ has the \textit{endomorphism closure property} (ECP) if for any $f \in F_n$ and all $\varphi_1, \ldots, \varphi_n \in \text{End}(A)$, the induced mapping $f^*(\varphi_1, \ldots, \varphi_n)$ is again an endomorphism of $A$. A variety $V$ of algebras has ECP if all algebras from $V$ have ECP.

\textbf{Definition 2.3} ([8]). Let $A = \langle A, F \rangle$ be an algebra. We say that $A$ has the \textit{retraction closure property} (RCP) if the set $\text{Retr}(A)$ is closed under induced mappings of $A$, i.e., for any $f \in F_n$ and all $\varphi_1, \ldots, \varphi_n \in \text{Retr}(A)$,