Testing for edge terms is decidable

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Abstract. This paper defines a computational problem, the edge-like problem, and proves that the problem is a decidable one when the input sets are finite. The edge-like problem is relevant to the field of universal algebra as it is a common generalization of several problems currently of interest in that field, and this paper proves that several of these problems are decidable.

1. Introduction

In this paper, we examine a particular class of conditions that individually may or may not hold true for a given algebra, and we prove that satisfaction of such a condition is a decidable proposition. This paper generalizes the work of Maróti in [9] and the proof of this paper’s main result proceeds along very similar lines to those of that paper.

In Section 2, we introduce some notation needed to simplify the proofs in this paper, and define the class of conditions that we will be examining. In Section 3, we define the characteristic triple of an operation, a partial evaluation of that operation which is compatible with the condition under examination. We then define what it means to compose a function with characteristic triples, and prove that this notion of composition is compatible with ordinary functional composition. In Section 4, we define weak near unanimity operations and examine the properties of their characteristic triples. Weak near unanimity operations form the starting point for our ultimate decision procedure for solving our original condition, and we prove that they can serve this purpose. In Section 5, we introduce a partial order on characteristic triples and prove that order filters constructed in a specific way are computable. We then prove that an operation satisfying our original condition will have a minimal characteristic triple with respect to this partial order, and that we can search for such an operation in finite time. In Section 6, we examine three corollaries of the main result that have consequences for the field of Universal Algebra, one of which is in fact Maróti’s result on which this paper is based, namely the main result of [9].
2. Preliminaries

Let us begin with some simplifying notation and terminology that will be in use throughout this paper.

**Definition 2.1.**  
- Let \( O_A \) be the set of all operations on set \( A \).
- Given any set \( F \) of operations, let \( F^{(n)} \) denote the set of \( n \)-ary operations in \( F \).
- If \( M \) is a matrix, let \( M_i \) denote the \( i \)th row of \( M \) and let \( M^j_i \) denote the \( i \)th column of \( M \). Let \( M^j_i \) denote the entry of \( M \) in row \( i \), column \( j \).
- Let \( \omega^+ \) be the set that contains all finite ordinals and \( \omega \), the smallest infinite ordinal.
- Given \( g \in O_A^{(k)} \), \( g' \in O_A^{(\ell)} \) and \( n \geq 0 \), say that \( g' \) is an \( n \)-extension of \( g \) if there is an injection \( \sigma: \{0,\ldots,k-1\} \to \{0,\ldots,\ell-1\} \) such that
  - \( g'(a_0,\ldots,a_{\ell-1}) = g(a_{\sigma(0)},\ldots,a_{\sigma(k-1)}) \), and
  - \( \sigma \) restricted to \( \{0,\ldots,n-1\} \) is the identity function.

Now we can define the type of problem we will be examining.

**Definition 2.2.** An instance of the edge-like problem is a tuple of the form \( P = (A,F,M,S) \) where

1. \( A \) is a finite set,
2. \( F \) is a finite set of operations on \( A \),
3. \( S \subseteq A \),
4. \( M \) is an \( m \times n \) matrix with elements in \( \{x,y\} \) (consider the rows of \( M \) to be indexed by \( n - m \leq i < n \) and the columns indexed by \( 0 \leq j < n \)),
5. no two rows of \( M \) are equal,
6. \( M \) does not contain a row of all \( x \)'s or a row of all \( y \)'s, and
7. \( M \) does not contain a column of all \( x \)'s.

**Definition 2.3.**  
- Given an instance of the edge-like problem, let \( B \) denote the set of all binary functions \( f: S \times A \to A \).
- Given an instance of the edge-like problem and an operation \( f \in O_A^{(k)} \) for some \( k \geq n \) and \( i \geq n - m \) (notice that \( n - m \) may be negative) define the \( i \)th polymer of \( f \), \( f|_i \in B \) to be
  \[
  f|_i(x,y) = \begin{cases} 
  f(M_i,x^{k-n}) & \text{if } n - m \leq i < n, \\
  f(x^i y x^{k-i-1}) & \text{if } n \leq i < k, \\
  f(x^k) & \text{otherwise}.
  \end{cases}
  \]
- For simplicity of notation, define \( \nu = \{i : n - m \leq i < n\} \).
- The class \( EL \) (edge-like) is the class of all instances of the edge-like problem for which there is an idempotent \( f \in \langle F \rangle^{(k)} \) (the \( k \)-ary operations in the clone generated by \( F \)) for some \( k \geq n \) with \( f|_i(x,y) = x \) for all \( n - m \leq i \), all \( x \in S \), and all \( y \in A \). In this case, we will say that \( f \) witnesses \( P \)'s membership in \( EL \).