Dichotomy on intervals of strong partial Boolean clones

KARSTEN SCHÖLZEL

Abstract. The following result has been shown recently in the form of a dichotomy: For every total clone $C$ on $\mathbb{2} := \{0, 1\}$, the set $\mathcal{I}(C)$ of all partial clones on $\mathbb{2}$ whose total component is $C$ is either finite or of continuum cardinality. In this paper, we show that the dichotomy holds, even if only strong partial clones are considered, i.e., partial clones which are closed under taking subfunctions: For every total clone $C$ on $\mathbb{2}$, the set $\mathcal{I}_{\text{str}}(C)$ of all strong partial clones on $\mathbb{2}$ whose total component is $C$, is either finite or of continuum cardinality.

1. Introduction

First, let $A$ be an arbitrary finite set. Later we concentrate on the Boolean case, i.e., we let $A = \mathbb{2} := \{0, 1\}$.

A function $f : A^n \to A$ is called a total function on $A$. A function $f : S \to A$ with $S \subseteq A^n$ is called a partial function on $A$ and we denote the domain of $f$ by $\text{dom} f := S$. Let $\text{Op}(A)$ be the set of all total functions on $A$, and let $\text{Par}(A)$ be the set of all partial functions on $A$.

The function $e^n_i : A^n \to A$ defined by $e^n_i(x_1, \ldots, x_n) := x_i$ is called the $n$-ary projection onto the $i$-th coordinate. For each $a \in A$, the constant function $c^n_a : A^n \to A$ is defined by $c^n_a(x) = a$ for all $x \in A^n$.

Let $f \in \text{Par}(A)$ be $n$-ary and let $g_1, \ldots, g_n \in \text{Par}(A)$ be $m$-ary. The composition $F := f(g_1, \ldots, g_n)$ is an $m$-ary partial function defined by

$$F(x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

and

$$\text{dom} F := \left\{ x \in \bigcap_{i=1}^n \text{dom} g_i \mid (g_1(x), \ldots, g_n(x)) \in \text{dom} f \right\}.$$ 

Let $C \subseteq \text{Par}(A)$. Then $C$ is called a partial clone if it is composition closed and contains the projections. If, additionally, $C$ contains only total functions, i.e., $C \subseteq \text{Op}(A)$, then $C$ is a total clone.

Let $f, g \in \text{Par}(A)$. We say that $f$ is a restriction (or subfunction) of $g$, written $f \leq g$, if $\text{dom} f \subseteq \text{dom} g$ and $f(x) = g(x)$ for all $x \in \text{dom} f$. Let $X \subseteq \text{Par}(A)$. Then the strong closure of $X$, written $\text{Str}(X)$, is defined by

$$\text{Str}(X) := \{ f \in \text{Par}(A) \mid \exists g \in X : f \leq g \}.$$
If $X = \text{Str}(X)$, then $X$ is called strong, or restriction closed. Thus, a set $X$ of partial functions is strong if it contains all subfunctions of its functions, i.e., $f \in C$ for all $f \in \text{Par}(A)$ and $g \in C$ with $f \leq g$.

Let $\text{Rel}^{(h)}(A)$ be the set of all $h$-ary relations on $A$ for some $h \geq 1$, i.e., $\text{Rel}^{(h)}(A) := \{X \mid X \subseteq A^h\}$. Furthermore, let $\text{Rel}(A) := \bigcup_{h \geq 1} \text{Rel}^{(h)}(A)$.

Let $\varrho \in \text{Rel}^{(h)}(A)$, and let $f : S \to A$ with $S \subseteq A^n$ be an $n$-ary partial function. Then $f$ preserves $\varrho$ iff $f(M) \in \varrho$ for any $h \times n$ matrix $M = (m_{ij})$ whose rows belong to the domain of $f$, i.e., $(m_{i1}, \ldots, m_{in}) \in \text{dom}f$ for all $i$, and whose columns belong to $\varrho$.

Let $\text{pPol} R$ be the set of all partial functions preserving every relation $\varrho \in R$. Let $\text{Pol} R := (\text{pPol} R) \cap \text{Op}(A)$ be the set of all total functions preserving every relation $\varrho \in R$.

There are three different types of intervals which we consider here. Let $C$ be a total clone of $\text{Op}(A)$. Then we define the three intervals $\mathcal{I}(C)$, $\mathcal{I}_{\text{Str}}(C)$, and $\mathcal{I}_{\text{Str}}^≤(C)$ by

\[
\mathcal{I}(C) := \{X \subseteq \text{Par}(A) \mid X \text{ partial clone, and } C = X \cap \text{Op}(A)\},
\]

\[
\mathcal{I}_{\text{Str}}(C) := \{X \subseteq \text{Par}(A) \mid X \text{ strong partial clone, and } C = X \cap \text{Op}(A)\},
\]

\[
\mathcal{I}_{\text{Str}}^≤(C) := \{X \subseteq \text{Par}(A) \mid X \text{ strong partial clone, and } C \subseteq X\}
\]

= $\bigcup \{\mathcal{I}_{\text{Str}}(D) \mid D \text{ is a total clone and } C \subseteq D\}$.

Clearly, $\mathcal{I}_{\text{Str}}(C) \subseteq \mathcal{I}(C)$ holds.

We need the following families of total Boolean clones. Notice that every total Boolean clone can be written as the intersection of some clones in the list below (see Figure 1 and Section 3.1 [12]).

\[
T_a = \text{Pol}\{a\} \text{ for } a \in \{0, 1\},
\]

\[
T_{a,\mu} = \text{Pol}(\{(0, 1)^\mu \setminus \{b, \ldots, b\}\}) \text{ for } b \in \{0, 1\}, b \neq a.
\]

\[
T_{a,\infty} = \bigcap_{\mu \geq 2} T_{a,\mu} \text{ for } a \in \{0, 1\}.
\]

\[
M = \text{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ (set of all monotone functions)}
\]

\[
S = \text{Pol}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (the set of all self-dual functions)}.
\]

\[
L = \text{Pol}\{(x, x, y, y), (x, y, x, y), (x, y, y, x) \mid x, y \in \{0, 1\}\}
\]

(\text{the set of all linear functions}).

\[
\Lambda = \text{Clone}\{\land, c_0, c_1\}.
\]

\[
\Lambda = \text{Clone}\{\lor, c_0, c_1\}.
\]

\[
\Omega_1 = \text{Clone}(\text{Op}(1)(2)).
\]

The finite intervals $\mathcal{I}(C)$ have been described in [1, 13, 15–17], and the finite intervals of the form $\mathcal{I}_{\text{Str}}^≤(C)$ have been described in [5, 8]. These results can