Key (critical) relations preserved by a weak near-unanimity function

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Abstract. In the paper, we introduce a notion of a key relation, which is similar to the notion of a critical relation introduced by Keith A. Kearnes and Ágnes Szendrei. All clones on finite sets can be defined by only key relations. In addition, there is a nice description of all key relations on 2 elements. These are exactly the relations that can be defined as a disjunction of linear equations. In the paper, we show that in general, key relations do not have such a nice description. Nevertheless, we obtain a nice characterization of all key relations preserved by a weak near-unanimity function. This characterization is presented in the paper.

1. Introduction

The main result in clone theory is apparently the description of all clones on 2 elements obtained by E. Post in [9, 10]. Nevertheless, it seems unrealistic to describe all clones on bigger sets. For example, we know that we have a continuum of them. Also, we have many results that prove that the lattice of all clones is not only uncountable, but very complicated.

It turned out that uncountability is not crucial; for example, in [16], the lattice of all clones of self-dual operations on 3 elements was described, even though this lattice has continuum cardinality. The main idea of that paper and many other papers in clone theory is an accurate work with relations. The fact that we have known all maximal clones for 45 years [12] and still do not have any description of all minimal clones just proves that working with relations is much easier than with operations.

We have $2^{|A|^n}$ relations of arity $n$ on a set $A$, which is a huge number even for $|A| = 4$ and $n = 3$. But if we check most of the significant papers in clone theory, we will see that all the relations arising there have a nice characterization: they are symmetric or have some regular structure. In this paper, we will try to provide a mathematical background to this observation.

First, it is easy to notice that we do not need relations that can be represented as a conjunction of relations with smaller arities [15, 16]. Relations that cannot be represented in this way are called essential. Second, observe
that if a relation is an intersection of other relations from the relational clone, then we do not need this relation to define this relational clone. Relations that cannot be represented in this way are called maximal in [16] and critical in [5].

It turned out that all critical relations $\rho \subseteq A^h$ have the following property: there exists a tuple $\beta \in A^h \setminus \rho$ such that for every $\alpha \in A^h \setminus \rho$, there exists a unary vector-function $\Psi = (\psi_1, \ldots, \psi_h)$ which preserves $\rho$ and gives $\Psi(\alpha) = \beta$. This means that every tuple which is not from $\rho$ can be mapped to $\beta$ by a vector-function preserving $\rho$. A relation satisfying this property is called a key relation, and a tuple $\beta$ is called a key tuple for this relation.

This property seems to be profitable because it is a combinatorial property of a relation which does not involve any difficult objects (no clones, no relational clones, no primitive positive formulas). Another motivation to study key relations is a nice description of all key relations on 2 elements. These are exactly the relations that can be defined as a disjunction of linear equations.

As we show in the paper, key relations on bigger sets can be complicated. But it turned out that we can get a very similar characterization of key relations if they are preserved by a weak near-unanimity function (WNU). In this case, we show that all the variables of the relation can be divided into two groups, and the relation can be divided into two parts. The first part is very similar to the relation $\{a, b\}^n \setminus \{a\}^n$, and the second part can be defined by a linear equation in some abelian group.

The consideration of key relations preserved by a WNU seems to be justified for the following reason. First, let us consider an algebra with all the operations from a clone. We know that if we have an idempotent algebra $A$ without a weak near-unanimity term, then we can find a factor of $A$ whose operations are essentially unary, where a factor is a homomorphic image of a subalgebra of $A$ [2, 8]. This means that if a relational clone is not preserved by a WNU, then we can find relations in it which are as complicated as in general, i.e., in a relational clone of all relations on a finite set. To show this, we need to consider the idempotent reduction of the corresponding clone, and then the corresponding factor. Thus, if we cannot describe all key relations, then we need to consider relational clones preserved by a WNU.

Second, the importance of a WNU was discovered while studying the constraint satisfaction problem. The standard way to parameterize interesting subclasses of the constraint satisfaction problem is via finite relational structures [3, 4]. The main problem is to classify those subclasses that are tractable (solvable in polynomial time) and those that are NP-complete. It was conjectured that if a core of a relational structure has a WNU polymorphism, then the corresponding constraint satisfaction problem is tractable, otherwise it is NP-complete [1, 2]. We believe that this characterization can be helpful in proving this conjecture.

The paper is organized as follows. In Sections 2 and 3, we give necessary definitions and formulate the main results of the paper. That is, a description of all key relations on 2 elements (with a proof) and a characterization of