Injective hulls are not natural

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Abstract. In a category with injective hulls and a cogenerator, the embeddings into injective hulls can never form a natural transformation, unless all objects are injective. In particular, assigning to a field its algebraic closure, to a poset or Boolean algebra its MacNeille completion, and to an \( R \)-module its injective envelope is not functorial, if one wants the respective embeddings to form a natural transformation.

1. Introduction

Projectivity and injectivity are fundamental concepts of modern mathematics. The question whether a given category has enough injectives (so that every object may be embedded into an injective one) or even injective hulls (so that such embeddings may be chosen to be essential), as well as the dual questions (enough projectives, projective covers), have been investigated for many categories, particularly in commutative and homological algebra, algebraic geometry, topology, and in functional analysis.

Existence of enough injectives is often facilitated by the existence of an injective cogenerator in the given category, in which case it is easy to see that, for each object \( A \), the embedding \( \iota_A : A \to EA \) into an injective object \( EA \) may be chosen naturally, so that \( E \) becomes an endofunctor of \( C \) and \( \iota : \text{Id}_C \to E \) a natural transformation. Under fairly mild additional conditions, the existence of an injective cogenerator even gives injective hulls, and the question then is again: are they natural? Although injective hulls are uniquely determined, up to isomorphism, the somewhat surprising answer that we give in this paper is: \textit{never}, unless the situation was trivial, in the sense that all objects were injective, in which case the injective-hull functor is given by the identity functor.

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2. Preliminaries

Throughout the paper, $\mathcal{H}$ is an arbitrary class of morphisms in a category $\mathcal{C}$. Although classically one thinks of the morphisms in $\mathcal{H}$ as “embeddings”, there is no a-priori assumption on $\mathcal{H}$. At first we recall some known definitions in order to fix our terminology.

Definitions 2.1. (1) An object $I$ of $\mathcal{C}$ is $\mathcal{H}$-injective if the function $\mathcal{C}(h, I): \mathcal{C}(B, I) \to \mathcal{C}(A, I)$ is surjective for every $h: A \to B$ in $\mathcal{H}$ (so that every $f: A \to I$ “extends” to some $g: B \to I$ with $g \cdot h = f$). The $\mathcal{H}$-injective objects form the full subcategory $\mathcal{H}$-$\text{Inj}$ of $\mathcal{C}$.

(2) A morphism $h$ in $\mathcal{H}$ is $\mathcal{H}$-essential if for every morphism $g$, the composite $g \cdot h$ lies in $\mathcal{H}$ only if $g$ does. The class of all $\mathcal{H}$-essential morphism in $\mathcal{C}$ is denoted by $\mathcal{H}^*$.

(3) $\mathcal{C}$ is said to have enough $\mathcal{H}$-injectives if for every object $A$ in $\mathcal{C}$ there is a morphism $\iota_A: A \to EA$ in $\mathcal{H}$ with an $\mathcal{H}$-injective object $EA$; if, in addition, $\iota_A$ can be chosen to be $\mathcal{H}$-essential, then $\mathcal{C}$ has $\mathcal{H}$-injective hulls (often also called envelopes). If, in either case, $E$ extends to an endofunctor of $\mathcal{C}$ making $\iota$ a natural transformation, we shall say that $\mathcal{C}$ has naturally enough $\mathcal{H}$-injectives or that $\mathcal{H}$-injective hulls are natural, respectively.

(4) A class $\mathcal{G}$ of objects is cogenenerating (also coseparating) in $\mathcal{C}$ if for any two distinct morphisms $u, v: X \to A$ in $\mathcal{C}$ there is a morphism $h: A \to G$ with $G \in \mathcal{G}$ and $h \cdot u \neq h \cdot v$; equivalently, if for every object $A$ in $\mathcal{C}$, the source of all morphisms with domain $A$ and codomain in $\mathcal{G}$ is monic. For $\mathcal{G}$ small, this is the same as to say that the canonical morphism

$$\iota_A: A \to \prod_{G \in \mathcal{G}} G^{\mathcal{C}(A, G)}$$

is a monomorphism, provided that the needed products exist in $\mathcal{C}$.

(5) By an $\mathcal{H}$-cogenenerator in $\mathcal{C}$ we mean a (small) set $\mathcal{G}$ of objects in $\mathcal{C}$ such that all small-indexed products of $\mathcal{G}$-objects exist in $\mathcal{C}$ and that the canonical morphisms $\iota_A$ of (4) lie in $\mathcal{H}$. $\mathcal{G}$ is an $\mathcal{H}$-injective $\mathcal{H}$-cogenenerator if, in addition, all objects in $\mathcal{G}$ are $\mathcal{H}$-injective.