Root closure in algebraic orders

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Abstract. We obtain a characterization of root closed algebraic orders by means of their conductor. It provides the root closure of an algebraic order. Actually, non-integrally closed root closed orders are exceptional. In the same way, we study \( n \)-root closedness of algebraic orders, for a given integer \( n \).

1. Introduction and notation. Let \( R \) be an integral domain with quotient field \( K \) and \( n \geq 1 \) an integer. We say that:

- \( R \) is \( n \)-root closed, if whenever \( x \in K \), \( x^n \in R \), then \( x \in R \).
- \( R \) is \((A-)root closed if \( R \) is \( n \)-root closed for all integers \( n \geq 1 \) (\( n \in A \), for \( A \) a nonempty subset of \( \mathbb{N}^+ \)).
- \( R \) is seminormal if whenever \( x \in K \), \( x^2, x^3 \in R \), then \( x \in R \).
- \( R \) is t-closed if whenever \( x \in K \), \( r \in R \), \( x^2 - rx, x^3 - rx^2 \in R \), then \( x \in R \).

Of course, an integrally closed integral domain is \( n \)-root closed and t-closed, and, for each \( n \geq 2 \), an \( n \)-root closed or a t-closed integral domain is seminormal; the converses do not hold in general. Let \( R \) be a subring of a ring \( S \) and \( n \geq 1 \) an integer. In the same way, we say that \( R \) is \( n \)-root closed in \( S \) if whenever \( x \in S \), \( x^n \in R \), then \( x \in R \), and \( R \) is root closed in \( S \) if \( R \) is \( n \)-root closed in \( S \) for all integers \( n \geq 1 \). All these closedness properties commute with localization. For t-closedness, see [5] and [6].

The \( n \)-root closure (resp. root closure, seminormalization, t-closure) of an integral domain \( R \) is the smallest \( n \)-root closed (resp. root closed, seminormal, t-closed) overring of \( R \) (it exists). Similarly, we define the \( n \)-root closure, root closure, seminormalization, t-closure of \( R \) in an overring \( S \).

Let \( K \) be a number field and \( \mathcal{O}_K \) its ring of integers. A subring of \( \mathcal{O}_K \) with quotient field \( K \) is called an (algebraic) order in \( K \). A local order is the localization of an order \( R \) at a prime ideal of \( R \).

There is a lot of literature about root closure in integral domains (cf. [3], [4], [7] and the survey of D. F. Anderson [2]). Curiously, root closure in algebraic orders has not been studied in the general case, although results of G. Angermüller are not far from a characterization of root closedness.

We begin by characterizing root closed orders in Section 2. Let \( R \) be a non-integrally closed order, with integral closure \( \bar{R} \). Then, \( R \) is root closed if and only if \( R \) is seminormal.

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and for any \( P \in \text{Max}(\bar{R}) \) containing the conductor of \( R \to \bar{R} \), the residue field \( \bar{R}/P \) has two elements. This result leads to the construction of the root closure of a non-integrally closed order. Let \( R \) be a non-integrally closed order, with integral closure \( \bar{R} \). The root closure of \( R \) is \( R + \cap P \), where \( \cap P \) is the set of all \( P \in \text{Max}(\bar{R}) \) containing the conductor of \( R \to \bar{R} \) and such that \( \bar{R}/P \) has two elements. This shows that in most cases, the root closure and integral closure coincide.

Section 3 is devoted to the study of \( n \)-root closedness, for a given integer \( n \geq 1 \). In fact, results are more involved than for root closedness and need conditions on \( t \)-closedness.

A generalization of these results can be done by considering residually finite one-dimension-
Noetherian integral domains with finite integral closure.

For a commutative ring \( R \) and an ideal \( I \) in \( R \), we denote by \( \text{V}_{\bar{R}}(I) \) the set of prime ideals in \( R \) containing \( I \). For an integral domain \( R \), we denote by \( \bar{R} \) its integral closure, by \( N(I) \) its root closure and by \( N_n(I) \) its \( n \)-root closure, for \( n \in \mathbb{N}^* \). The conductor of \( R \to \bar{R} \) is called the conductor of \( R \). If \( S \) is a finite set, the number of elements of \( S \) is denoted by \( |S| \). As usual, \( \mathbb{N}^* \) is the set of all nonzero natural numbers and \( \mathbb{F}_q \) is the finite field with \( q \) elements, \( q \) a power of a prime integer.

2. Root closure. We begin by recalling some results which will be used throughout this paper.

– [5, Proposition 4.9 and Corollaire 4.13] Let \( R \) be an algebraic order. Then \( R \) is seminormal (resp. \( t \)-closed) if and only if the conductor of \( R \) is a radical ideal in \( \bar{R} \) (resp. the conductor of \( R \) is a radical ideal in \( \bar{R} \) and the spectral map \( \text{Spec}(R) \to \text{Spec}(\bar{R}) \) is bijective).

– [3, Theorem 1.7] Let \( A \) and \( B \) be rings with a common ideal \( I \) and \( S \) a nonempty subset of \( \mathbb{N}^* \). Then \( A \) is \( S \)-root closed in \( B \) if and only if \( A/I \) is \( S \)-root closed in \( B/I \).

– [4, Theorem 1] Let \( R \) be a Noetherian domain of dimension one such that the integral closure \( S \) of \( R \) is a finite \( R \)-module and \( S/M \) is a finite field with at least 3 elements for every \( M \in \text{Max}(S) \). If \( R \) is root closed, then \( R \) is integrally closed.

In fact, this last theorem provides a characterization of root closed orders and we mimic its proof.

**Theorem 2.1.** Let \( R \) be a non-integrally closed algebraic order with conductor \( I \). Then \( R \) is root closed if and only if the two following conditions are satisfied.

1. \( R \) is seminormal.
2. \(|\bar{R}/P| = 2 \) for every \( P \in \text{V}_{\bar{R}}(I) \).

**Proof.** Let \( R \) be a non-integrally closed root closed algebraic order with conductor \( I \). Then \( R \) is seminormal and \( I \) is a radical ideal in \( R \) and \( \bar{R} \). It follows that \( I = \bigcap_{M \in \text{V}_{\bar{R}}(I)} M \).

Let \( M \in \text{V}_{\bar{R}}(I) \). Then \( MR_M \) is the conductor of \( R_M \). Moreover, \( R_M \) is root closed and non-integrally closed. Let \( P \in \text{V}_{\bar{R}}(I) \) lying over \( M \) such that \( |\bar{R}/P| > 2 \) and set \( n = |\bar{R}/P| - 1 > 1 \). Then \( \bar{R}/P \cong R_M/P_M \) is a finite field with \( n + 1 \) elements which contains a primitive \( n \)th root of unity. By [4, Main Lemma], \( R_M \) is a local ring. Thanks to [4, Theorem 1], \( R_M \) is integrally closed, a contradiction and (2) is gotten.

Now assume that conditions (1) and (2) are fulfilled. Since \( R \) is seminormal, \( I \) is a radical ideal in \( R \) and \( \bar{R} \). For \( M \in \text{V}_{\bar{R}}(I) \), denote by \( R' \) the localizations at \( M \) of \( R \) and \( M \). Then \( M' \) is