Annular capillary surfaces

By

ALAN ELCRAT, TAE-EUN KIM and RAY TREINEN

Abstract. The meniscus in a symmetric annular capillary tube is investigated. The contact angles on the inner and outer tube surface need not be the same. Existence and qualitative properties of solutions are obtained using an iteration similar to that used by Johnson and Perko in the case of a circular capillary tube. If the contact angles have the same sign ideas of Siegel are used to give asymptotic estimates using circular arcs as comparison curves.

1. Introduction. The equilibrium shapes of liquid-air interfaces have been the object of study for two hundred years. The mathematical description of these surfaces is given in terms of the Young-Laplace equation

$$\text{div} \left( \frac{\text{grad} \ u}{W} \right) = \kappa u,$$

where $W = (1 + |\text{grad} \ u|^2)^{1/2}$ and $\kappa = \rho \sigma / g$ is the “capillary constant”, where $\rho$ is the density of the liquid, $\sigma$ is the surface tension at the surface of the liquid, and $g$ is the gravitational constant. The solutions of this equation and their properties have been of great interest to mathematicians, and general existence theory, state of the art numerical algorithms, and asymptotic methods have been applied. On the other hand, in contrast to the case of minimal surfaces, explicit solutions are not available even in the simplest geometries. In the fundamental case of a circular capillary tube knowledge is nearly complete. It is therefore surprising that a related special geometry, that of an “annular” tube, has not been extensively studied. There are some numerical and approximate results available ([4], [5], [6], [9]), but theoretical results known are mainly what can be derived from those for a general geometry. It is our purpose here to make a contribution to understanding this problem.

We consider here capillary surfaces bounded by two concentric circular cylindrical walls. The base domain is a circular annulus and we impose (possibly different) constant contact angles on each circle.

Mathematics Subject Classification (2000): 76B45, 49Q10, 35J60, 35J65.
Known results imply existence and uniqueness of a solution of the implied nonlinear boundary value problem for the capillary surface equation ([1], [10]), but we study directly the ordinary differential equation boundary value problem for axisymmetric solutions. The methods that we use are closely related to those that have been used for circular cross section capillary tubes ([1, Ch. 2, 3]), and in the process of dealing with this problem directly we obtain a number of interesting properties of solutions. The existence proof uses techniques introduced by Johnson and Perko for the circular capillary tube. We first deal with a “half tube” problem in which the surface is horizontal at one end of its domain using [3], and then piece these together for the solution of the original problem. A simple computational algorithm is presented for solving the problem numerically which is based on the existence proof ideas. This has been implemented with a MATLAB code and numerical examples are presented.

In addition, we have obtained estimates for annular surfaces using circular arcs for comparison in the merideanal plane. Here we use the ideas of Siegel [8], in which the monotonicity of longitudinal curvature plays an essential role.

2. An existence theorem. The problem that we consider takes the form

\( \left( \frac{ru'}{\sqrt{1 + (u')^2}} \right)' = \kappa ru, \quad 0 < a < r < b, \)  
\( u'(a) = -\cot(\gamma_a), \quad u'(b) = \cot(\gamma_b), \)

where \( r \) is the radial coordinate, \( u \) is the surface height, and \( \gamma_a, \gamma_b \) are the contact angles at \( r = a, r = b \), respectively, and \( \kappa = \rho g / \sigma \), where \( \rho \) is the fluid density, \( g \) the gravitational constant, and \( \sigma \) the surface tension. We assume that \( 0 \leq \gamma_a, \gamma_b < \frac{\pi}{2} \) in what follows. The other combinations of possibilities will be dealt with later in the paper.

Without loss of generality we may introduce a scaling so that \( \kappa = 1 \).

Since we have assumed \( 0 \leq \gamma_a, \gamma_b < \frac{\pi}{2} \) it is reasonable to assume that there is a unique \( m, a < m < b \), at which \( u'(m) = 0 \). We take this as a starting point and consider the auxiliary problem

\( \left( \frac{ru'}{\sqrt{1 + (u')^2}} \right)' = ru, \quad r > m, \)  
\( u(m) = h, \quad u'(m) = 0, \)

where we assume \( h > 0 \). This problem is equivalent to the system

\( v(r) = \frac{1}{r} \int_m^r u(\xi) \xi \, d\xi \)