Embedding of non-commutative $L^p$-spaces: $p < 1$

By

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Abstract. If $(\mathcal{N}, \tau)$ is a finite von Neumann algebra and if $(\mathcal{M}, \nu)$ is an infinite von Neumann algebra, then $L_p(\mathcal{M}, \nu)$ does not Banach embed in $L_p(\mathcal{N}, \tau)$ for all $p \in (0, 1)$. We also characterize subspaces of $L_p(\mathcal{N}, \tau)$, $0 < p < 1$ containing a copy of $l_p$.

0. Introduction. In general, the development of (quasi) Banach space theory has clearly shown that there is no hope left for a complete structural theory of (quasi) Banach spaces, although one can still hope to have such a theory in some special cases. One such case is given by the family of non-commutative $L_p$-spaces associated with hyperfinite von Neumann algebras. The theory of non-commutative $L_p$-spaces began from the paper of von Neumann [N], where he introduced (matricial, finite dimensional) non-commutative $L_p$-spaces ($C^n_p$ in the modern notation, with the index $n$ reflecting the dimension of the matrix space) and briefly commented on the (isometric) difference between finite dimensional $L_p$-spaces and their matrix counterparts. Of course, the classical Banach spaces $L_p(0, 1)$ and $l_p$ belong to this family and the relationship between these two classes of spaces is already well documented in the Banach’s book [1].

Recently, the embedding properties of non-commutative $L_p$-spaces have been extensively studied (cf. [3], [4, 5], [9–11] and [13,14]). In particular, it was shown in [3] and [13,14] that for any $1 \leq p < \infty$ and $p \neq 2$, the $L_p$-space associated with a semifinite von Neumann algebra which has a direct summand of type $I_\infty$ (respectively, $II_\infty$) can not be embedded into the $L_p$-space associated with a finite von Neumann algebra (respectively, with a semifinite von Neumann algebra without a direct summand of type $II_\infty$). As a consequence, with the same condition on $p$, the Schatten class $C_p$ (i.e. the non-commutative $L_p$-space associated with the von Neumann algebra of all bounded linear operators on infinite dimensional Hilbert space) is not isomorphic to a subspace of $L_p(\mathcal{N})$ for any finite von Neumann algebra $\mathcal{N}$. These results play an important rôle in the Banach-isomorphic classification of the non-commutative $L_p$-spaces associated with hyperfinite and semifinite von Neumann algebras in [14] and [3]. The paper [3] also contains many related results of general interest, in particular

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it presented a description of subspaces of non-commutative $L_p$-spaces containing a copy of the space $l_p$. This note deals with these properties in the case of $0 < p < 1$. We will show that the results asserting non-embedding for various pairs of non-commutative $L_p$-spaces, $1 \leq p < 2$ from [3] can be extended to the case $0 < p < 1$. Although our arguments depend very much on those from [3], we provide significant simplifications and additional insights into the arguments which seem essential for the case $0 < p < 1$.

Throughout this note we will use the same terminology and notation as in [3]. In the sequel, $\mathcal{N}$ will always denote a von Neumann algebra equipped with a normal faithful normalized atomless trace $\tau$. The assumption that $\tau$ is atomless causes no loss of generality (cf. [3], Section 2) and all non-embedding results presented in this note hold also without this assumption. By $L_p(\mathcal{N}, \tau)$, or simply, $L_p(\tau)$ we denote the associated non-commutative $L_p$-space (see e.g. [2]). Let $x \in L_p(\tau)$ and $\delta > 0$. We recall that the $p$-modulus of $x$ is

$$\omega_p(x, \delta) = \sup \{ \|xe\|_p : e \in \mathcal{P}(\mathcal{N}), \tau(e) \leq \delta \},$$

where $\mathcal{P}(\mathcal{N})$ denotes the lattice of the projections of $\mathcal{N}$. For simplicity, we refer to number $\omega(x, \delta) = \omega_1(x, \delta)$ as the modulus of $x$. Note that (cf. [3])

$$\omega_p(x, \delta) = \omega_p(x^*, \delta) = \omega_p(|x|, \delta) = (\omega(|x|^p, \delta))^{\frac{1}{p}} = \left( \int_0^\delta \mu_t(x)^p dt \right)^{\frac{1}{p}},$$

where, as usual, $t \mapsto \mu_t(x)$ stands for the generalized singular number function of $x$. A subset $K \subset L_p(\tau)$ is said to be uniformly $p$-integrable if

$$\lim_{\delta \to 0} \sup_{x \in K} \omega_p(x, \delta) = 0.$$

1. An embedding result. The following is the extension to the case $0 < p < 1$ of the main Theorem 4.2 in [3]. Let $(x_{ij})_{i,j \geq 1} \subset L_p(\tau)$ be a bounded matrix (i.e. $\sup_{i,j} \|x_{ij}\|_p < \infty$). Recall that a generalized diagonal of $(x_{ij})_{i,j \geq 1}$ is a sequence $(x_n)_{n \geq 1} = (x_{i_n,j_n})_{n \geq 1}$ with $i_1 < i_2 < \ldots$ and $j_1 < j_2 < \ldots$.

**Theorem 1.1.** Let $0 < p < 1$. Let $(x_{ij})_{i,j \geq 1} \subset L_p(\tau)$ be a bounded matrix. Suppose that every generalized diagonal of $(x_{ij})$ is unconditional. Then one of the following three alternatives holds:

1. Some row or column contains a subsequence equivalent to the canonical basis of $l_p$.
2. There is a constant $\lambda > 0$ such that for all $n$ some row and some column contain finite subsequences $\lambda$-equivalent to the canonical basis of $l_p^n$.
3. There is a generalized diagonal $(x_n)_{n \geq 1}$ such that for every subsequence $(y_n)_{n \geq 1}$ of $(x_n)_{n \geq 1}$

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \left\| \sum_{k=1}^n y_k \right\|_p = 0.$$