Notes on commutators on Herz-type spaces

By

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Abstract. We show that if the commutator of Coifman, Rochberg and Weiss \([b, T]\) is bounded on the Herz spaces, then \(b\) is a CMO function. We also show the factorization theorem for some Herz-type Hardy spaces.

1. Introduction. Let \(b\) be a locally integrable function on \(\mathbb{R}^n\) and let \(T\) be a Calderón-Zygmund singular integral operator (see Section 2.4). Consider the commutator operator \([b, T]\) defined by

\[
[b, T]f = b \cdot Tf - T(bf).
\]

Coifman, Rochberg and Weiss [2] proved that \([b, T]\) is a bounded operator on \(L^q(\mathbb{R}^n)\), \(1 < q < \infty\), if and only if \(b\) is a BMO function.

Grafakos, Li and Yang [6] showed that \([b, T]\) is bounded on the Herz space \(K^{\alpha,p}_q(\mathbb{R}^n)\) if \(b\) is a CMO function (for details, see Section 2 and 3). They proposed the problem whether the converse of their theorem is true.

In this paper we give a partial positive answer to this problem. To prove our theorem we show the factorization theorem for the Herz-type Hardy spaces.

Of course the converse is not true when \(T \equiv 0\), so we need to assume some nondegenerate condition for \(T\).

2. Definitions and Notations. The following notation is used: For a set \(E \subset \mathbb{R}^n\) we denote the Lebesgue measure of \(E\) by \(|E|\). We denote a characteristic function of \(E\) by \(\chi_E\).

We write a ball of radius \(r\) centered at \(x_0\) by \(B(x_0, r) = \{x; |x - x_0| < r\}\).

We define the Herz spaces, the Hardy spaces associated with the Herz spaces and CMO (see [1], [4], [5] and [10]).

Let \(1 < p, q < \infty\) and \(\alpha \in \mathbb{R}\).

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2.1. Herz space.

Definition 1. The homogeneous Herz space is defined by

\[ \dot{K}^{\alpha,p}_q(R^n) = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \| f \|_{\dot{K}^{\alpha,p}_q} < \infty \}, \]

where \( \| f \|_{\dot{K}^{\alpha,p}_q} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left( \int_{A_k} |f(x)|^q \, dx \right)^{p/q} \right\}^{1/p} \)

and \( A_k = \{ x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k \} \).

Now we define the Hardy spaces associated with the Herz spaces. First we define maximal function. Let \( \varphi \in \mathcal{S} \) be a fixed function such that \( \text{supp}(\varphi) \subset B(0,1) \) and \( \int \varphi(x) \, dx \neq 0 \), then we define

\[ f^+(x) = \sup_{t>0} |f \ast \varphi_t(x)|, \quad \text{where} \quad \varphi_t(x) = t^{-n} \varphi(x/t). \]

Definition 2. The Hardy space \( \dot{H}^{q}_{\dot{K}^q}(R^n) \) is defined by

\[ \dot{H}^{q}_{\dot{K}^q}(R^n) = \{ f \in L^1(R^n) : f^+ \in \dot{K}^{q(1-1/q),1}_q(R^n) \}, \]

and we define \( \| f \|_{\dot{H}^{q}_{\dot{K}^q}} = \| f^+ \|_{\dot{K}^{q(1-1/q),1}_q}. \)

Remark. In [5] and [10], \( \dot{H}^{q}_{\dot{K}^q} \) is denoted by \( \dot{H}^{A,q}_{A,1} \) and \( \dot{H}^{q(1-1/q),1}_q \) respectively.

2.2. Atom. Following García-Cuerva [4] and Lu and Yang [10], we define atoms on \( \dot{H}^{q}_{\dot{K}^q}(R^n) \).

Definition 3. A function \( a(x) \) is a central \( (\dot{H}^{q}_{\dot{K}^q}) \)-atom if there exists a ball \( B(0,r) \) such that the following conditions are satisfied

\[ \text{supp}(a) \subset B(0,r), \quad \| a \|_{L^q} \leq |B(0,r)|^{1/q-1}, \quad \int a(x) \, dx = 0. \]

Lemma 1 [4]. If a function \( a(x) \) is a central \( (\dot{H}^{q}_{\dot{K}^q}) \)-atom, then \( \| a \|_{\dot{H}^{q}_{\dot{K}^q}} \leq C_{n,q} \), where \( C_{n,q} \) is a positive constant depending only on \( n \) and \( q \).

García-Cuerva [4] obtained the following atomic decomposition theorem for \( \dot{H}^{q}_{\dot{K}^q} \).

Proposition 1. If \( f \in \dot{H}^{q}_{\dot{K}^q}(R^n) \) then \( f \) can be represented as \( f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \), where \( a_k \) is a central \( (\dot{H}^{q}_{\dot{K}^q}) \)-atom supported on \( B(0,2^k) \) and \( \sum_{k=-\infty}^{\infty} |\lambda_k| \approx \| f \|_{\dot{H}^{q}_{\dot{K}^q}}. \)