Functions that are both $g$- and $h$-additive

By

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Abstract. We characterise the functions $\varphi$, that are simultaneously $g$-additive and $h$-additive if the bases $g, h$ don’t divide each other: $\varphi$ is a linear combination of step–functions and functions that are periodic and constant on some subblocs.

1. Introduction and main result. Let $g \in \mathbb{Z}, \geq 2$ be fixed. Then every $n \in \mathbb{N}$ can be uniquely represented in the form

$$n = \sum_{r \geq 0} e_r(n) g^r, \quad \text{where} \quad e_r(n) \in \{0, 1, \ldots, g - 1\}.$$

Definition. A function $\varphi : \mathbb{N}_0 \to \mathbb{C}$ is called $g$-additive iff

$$\varphi \left( \sum_{r \geq 0} e_r(n) g^r \right) = \sum_{r \geq 0} \varphi(e_r(n) g^r), \quad \text{for all} \quad n \in \mathbb{N}_0$$

($g \in \mathbb{Z}, \geq 2$). These functions were first introduced and studied by Gelfond [1].

In this paper we investigate functions, that are $g$-additive and simultaneously $h$-additive. There are two types of examples.

Example 1. Let $g, h, k \in \mathbb{N}, \ g, h \geq 2$. We define the step–functions

$$\iota_k(n) := [g, h]^{k-1} \left[ \frac{n}{[g, h]^{k-1}} \right], \quad n \in \mathbb{N}_0.$$

Here and in the sequel $[g, h]$ is the least common multiple of $g$ and $h$; later $(g, h)$ will denote the greatest common divisor.

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Example 2. We call a function $\varphi : \mathbb{N}_0 \to \mathbb{C}$

- $p$-periodic ($p \in \mathbb{N}$) iff $\varphi(n + p) = \varphi(n)$, for all $n \in \mathbb{N}_0$;
- $q$-constant ($q \in \mathbb{N}$) iff $\varphi(aq + b) = \varphi(aq)$, for all $a, b \in \mathbb{N}_0, b < q$.

For $g, h, k \in \mathbb{N}$, we introduce the vector-space

$$V_k(g; h) := \{ \varphi : \mathbb{N}_0 \to \mathbb{C} : \varphi(0) = 0, \varphi \text{ is } (g, h)^k\text{-periodic and } [g, h]^{k-1}\text{-constant} \}.$$ 

Our main result is the following

**Theorem.** Let $g, h$ be integers $\geq 2$ and assume $g \nmid h, h \nmid g$. Then

1. every function $\varphi$, that is simultaneously $g$-additive and $h$-additive, can be uniquely represented in the form

$$\varphi = \sum_k c_k \iota_k + \sum_k \varphi_k,$$

where $c_k \in \mathbb{C}, \varphi_k \in V_k(g; h)$

and the star indicates, that we sum over the natural $k$ with the property $[g, h]^k \nmid (g, h)^k$.

2. The complex vector-space $V$ of all simultaneously $g$- and $h$-additive functions has dimension

$$\dim_C V = \sum_k \frac{(g, h)^k}{[g, h]^{k-1}}.$$ 

3. Every simultaneously $g$- and $h$-additive function is already $(g, h)$-additive, if $(g, h) > 1$.

**Remark.** We suppose that $\log g / \log h$ is irrational. Then there exist positive integers $l, m$ so that $g^l \nmid h^m, h^m \nmid g^l$. It follows from the theorem, that the space $V$ of all simultaneously $g$-additive and $h$-additive functions $\varphi$ has finite dimension and every $\varphi \in V$ is $(g^l, h^m)$-additive, if $(g^l, h^m) > 1$. If $k_0$ is chosen so that $(g^l, h^m)^{k_0} < (g^l, h^m)^{k-1}$, for all $k > k_0$, and $d := (g^l, h^m)^{k_0}$, then $\varphi(dn) = cn$ for every positive integers $n$. This was proved by Uchida [3, Theorem 1] by another method. If further $\varphi$ has the property $\varphi(eg^r) = \varphi(e), 0 \leq e < g, r \geq 0$ (“strongly $g$-additive”), then the theorem gives $\varphi = 0$, so every simultaneously strongly $g$-additive and $h$-additive function vanishes. This is a result of Toshimitsu [2, Theorem 3].

2. The functions $\iota_k$.

**Proposition 1.** For $g, h \in \mathbb{Z}, \geq 2, k \in \mathbb{N}$, $[g, h]^{k-1}(g, h)^k$, then $\iota_k$ is simultaneously $g$-additive and $h$-additive; if $(g, h) > 1, \iota_k$ is $(g, h)$-additive.