Frobenius $\mathbb{Q}$-groups

By

M. R. Darafsheh and H. Sharifi

Abstract. A finite group whose irreducible complex characters are rational is called a $\mathbb{Q}$-group. In this paper we will find the structure of Frobenius $\mathbb{Q}$-groups.

1. Introduction. Let $G$ be a finite group and $\chi$ a complex character of $G$. The field generated by all $\chi(x)$, $x \in G$, is denoted by $\mathbb{Q}(\chi)$. By definition a complex character $\chi$ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$, and a finite group $G$ is called a rational group or a $\mathbb{Q}$-group if every irreducible complex character is rational. Examples of $\mathbb{Q}$-groups are the symmetric group $S_n$ and the Weyl groups of the complex Lie algebras, see [2]. Finite $\mathbb{Q}$-groups have been studied by some mathematicians and there are some unsolved problems about them. It is shown in [4] that if $G$ is a solvable $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2, 3, 5\}$ where $\pi(G)$ denotes the set of prime divisors of $|G|$. Also in [3] it is proved that the only non-Abelian simple $\mathbb{Q}$-groups are the groups $SP_6(2)$ and $O^+_8(2)$. But classifying finite $\mathbb{Q}$-groups still remains an open research problem. In the book [6] several open problems have been raised concerning $\mathbb{Q}$-groups. In this note we will find the structure of Frobenius $\mathbb{Q}$-groups. In this paper all groups are finite and all characters are complex. The semi-direct product of groups $H$ and $K$ is denoted by $H \rtimes K$, and a cyclic group of order $n$ by $\mathbb{Z}_n$. Also, if $p$ is a prime number, $E(p^n)$ denotes the elementary Abelian $p$-group of order $p^n$. Finally, we write $Q_8$ for the quaternion group of order 8.

Before stating our main theorem, we will mention some well-known results about $\mathbb{Q}$-groups. An alternative characterization of $\mathbb{Q}$-groups is the following result which can be found in [5, p. 537, Satz 13.7].

Result 1. A group $G$ is a $\mathbb{Q}$-group if and only if for every $x \in G$ of order $n$ the elements $x$ and $x^m$ are conjugate in $G$, whenever $(m, n) = 1$. Equivalently, for each $x \in G$ we must have $N_G(\langle x \rangle)/C_G(\langle x \rangle) \cong \text{Aut}(\langle x \rangle)$.

Result 2. Quotients and direct products of $\mathbb{Q}$-groups are $\mathbb{Q}$-groups.

Mathematics Subject Classification (2000): 20C15.
Our main result is the following.

**Main Theorem.** If \( G \) is a Frobenius \( \mathbb{Q} \)-group, then exactly one of the following occurs:

1. We have \( G \cong E(3^n) : \mathbb{Z}_2 \), where \( n \geq 1 \) and \( \mathbb{Z}_2 \) acts on \( E(3^n) \) by inverting every non-identity element.
2. \( G \cong E(3^{2m}) : Q_8 \), where \( m \geq 1 \) and \( E(3^{2m}) \) is a direct sum of \( m \) copies of the 2-dimensional irreducible representation of \( Q_8 \) over the field with 3 elements.
3. \( G \cong E(5^2) : Q_8 \), where \( E(5^2) \) is the 2-dimensional irreducible representation of \( Q_8 \) over the field with 5 elements.

**2. Frobenius \( \mathbb{Q} \)-groups.** By definition a Frobenius group is a group \( G \) with a subgroup \( H \) such that \( 1 \neq H < G \) and \( H \cap H^x = 1 \) for all \( x \in G - H \). The subgroup \( H \) is called Frobenius complement and it is well-known that \( G \) has a normal subgroup \( K \), called Frobenius kernel, such that \( G = HK \), \( H \cap K = 1 \). To see this result and related results concerning Frobenius groups we refer the reader to [8]. In [1, p. 62] it is proved that solvable non-nilpotent \( \mathbb{Q} \)-groups with Sylow 2-subgroups isomorphic to the quaternion group of order 8 are Frobenius groups. This is a motivation for us to classify Frobenius \( \mathbb{Q} \)-group.

**Lemma.** Let \( G \) be a Frobenius group with complement \( H \). If \( G \) is a \( \mathbb{Q} \)-group, then \( H \cong \mathbb{Z}_2 \) or \( Q_8 \).

**Proof.** Let \( G \) be a Frobenius \( \mathbb{Q} \)-group with complement \( H \) and kernel \( K \). Then by Result 2, \( G/K \cong H \) is also a \( \mathbb{Q} \)-group. Since \( H \neq 1 \) and for any non-identity element \( x \) in \( H \) the elements \( x \) and \( x^{-1} \) are conjugate in \( H \), we have \( 2 \mid |H| \). We consider two cases.

Case (i) \( H \) is a non-solvable group.

Then by a result of Zassenhaus quoted in [8, p. 204], the group \( G \) has a normal subgroup \( G_0 \) with \( [G : G_0] \leq 2 \) such that \( G_0 \cong SL_2(5) \times M \) where \( M \) is a group of order prime to 2, 3 and 5. By [2] the group \( SL_2(5) \) is not a \( \mathbb{Q} \)-group, neither is any extension of \( SL_2(5) \) by an automorphism of order 2. Therefore this case does not lead to a \( \mathbb{Q} \)-group.

Case (ii). \( H \) is a solvable group.

Since \( H \) is a \( \mathbb{Q} \)-group by [4] we must have \( \pi(H) \subseteq \{2, 3, 5\} \), where \( \pi(H) \) denotes the set of prime divisors of \( |H| \). Since \( 2 \mid |H| \), hence by [8, p. 194], the Sylow 2-subgroups of \( H \) are either cyclic or generalized quaternion groups.

First assume that a Sylow 2-subgroup \( P \) of \( H \) is a generalized quaternion group. Then

\[
P = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} \rangle.
\]