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Realization of the Stasheff polytope

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Abstract. We propose a simple formula for the coordinates of the vertices of the Stasheff polytope (alias associahedron) and we compare it to the permutohedron.

Introduction. The Stasheff polytope $\mathcal{K}^n$, also called associahedron, appeared in the sixties in the work of Jim Stasheff [7] on the recognition of loop spaces. It is a convex polytope of dimension n with one vertex for each planar binary tree with $n+2$ leaves. There are various realizations of $\mathcal{K}^n$ as a polytope in the literature either published cf. [1], [2], [3], [8], [9], or unpublished (D. Grayson, M. Haiman). Here we propose a simple one, which has the following advantages, on top of being simple:

- it respects the symmetry,
- it fits with the classical realization of the permutohedron $\mathcal{P}^n$,
- the faces have simple equations.

To any planar binary tree we associate a point in the euclidean space by describing its coordinates (which are going to be positive integers) in terms of the structure of the tree. Explicitly the ith coordinate is the product of the number of leaves on the left side and the number of leaves on the right side of the ith vertex. The main idea is to start with the permutohedron and to think of it as the truncation of the standard simplex by some hyperplanes. Truncating only by the *admissible* hyperplanes gives the Stasheff polytope. From the explicit equations of the facets of the permutohedron we compute the coordinates of the intersections of the admissible hyperplanes and find the result mentioned above.

Convention. In the euclidean space $\mathbb{R}^n$ the coordinates of a point are denoted $x_1, \dots, x_n$. We denote by H the affine hyperplane whose equation is: $\sum_{i=1}^{i=n} x_i = \frac{1}{2}n(n+1)$. We adopt the notation $S(n) = \frac{1}{2}n(n+1)$.

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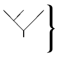
1. A simple realization of the Stasheff polytope. The *Stasheff polytope* $\mathcal{K}^n$ of dimension n (alias associahedron) is a finite cell complex whose k-cells are in bijection with the planar trees having $n-k+1$ internal vertices and $n+2$ leaves (so it is sometimes denoted K_{n+2}), cf. [7].

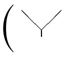
Let Y_n be the set of planar binary trees with $n+1$ leaves:

$Y_0 = \{|\}$, $Y_1 = \{$  $\}$, $Y_2 = \{$ ,  $\}$

$Y_3 = \{$ , , , ,  $\}$

The integer n is called the *degree* of $t \in Y_n$. We label the leaves of t from left to right by $0, 1, \ldots$. Then we label the internal vertices by $1, 2, \ldots$. The ith vertex is the one which falls in between the leaves $i-1$ and i. We denote by a_i, resp. b_i, the number of leaves on the left side, resp. right side, of the ith vertex. The product $a_i b_i$ is called the *weight* of the ith vertex. To the tree t in Y_n we associate the point $M(t) \in \mathbb{R}^n$ whose ith coordinate is the weight of the ith vertex:

$$M(t) = (a_1 b_1, \ldots, a_i b_i, \ldots, a_n b_n) \in \mathbb{R}^n.$$

For instance: $M($  $) = (1)$, $M($  $) = (1, 2)$, $M($  $) = (2, 1)$,
$M($  $) = (1, 2, 3)$, $M($  $) = (1, 4, 1)$.

Observe that the weight of a vertex depends only on the subtree that it determines.

We will show in the next section that all the points $M(t) \in \mathbb{R}^n$ for $t \in Y_n$ lie in the affine hyperplane H. The main point of this paper is the following result.

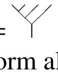
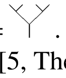
Theorem 1. *The convex hull of the points $M(t) \in \mathbb{R}^n$, for t a planar binary tree with $n+1$ leaves, is a realization of the Stasheff polytope $\mathcal{K}^{n-1}$ (alias associahedron) of dimension $n-1$.*

The proof will be given in the next section.

Let us recall the definition of the *permutohedron*. For any permutation s in the symmetric group S_n acting on the set $\{1, \ldots, n\}$, let $M(s) \in \mathbb{R}^n$ be the point with coordinates $M(s) = (s(1), \ldots, s(n))$. By definition the permutohedron $\mathcal{P}^{n-1}$ is the convex hull of the $n!$ points $M(s)$. Observe that the sum of the n coordinates of $M(s)$ is $S(n)$, hence all the points $M(s)$ lie in the affine hyperplane H.

Under interpreting a permutation as a planar binary tree with levels (cf. for instance [6]), and then forgetting the levels, one gets a well-defined map

$$\psi : S_n \twoheadrightarrow Y_n.$$

For instance $\psi(1\ 2\ 3) =$ , $\psi(1\ 3\ 2) = \psi(2\ 3\ 1) =$ . An algebraic interpretation of ψ in terms of dendriform algebras has been given in [5, Theorem 7.5].