A relation between the zeros of an $L$-function belonging to the Selberg class and the zeros of an associated $L$-function twisted by a Dirichlet character

By

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Abstract. Let $F(s)$ be a function belonging to the Selberg class. For a primitive Dirichlet character $\chi$, we can define the $\chi$-twist $F_\chi(s)$ of $F(s)$. If $F_\chi(s)$ also belongs to the Selberg class and satisfies some other conditions then there is a relation between the zeros of $F(s)$ and the zeros $F_\chi(s)$. Further we give an operator theoretic interpretation of this relation according to A. Connes’ study.

0. Introduction. Let $s = \sigma + it$ be a complex variable. The Riemann zeta function $\zeta(s)$ is defined by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

in the right-half plane $\sigma > 1$ and is continued holomorphically to the whole $s$-plane except for the simple pole $s = 1$. The study of the horizontal distribution of the zeros of $\zeta(s)$ is one of the most important problems in number theory. It is represented by the famous Riemann Hypothesis. If the Riemann Hypothesis (RH) is true, then one of the next natural problems for the Riemann zeta function is the study of the vertical distribution of the zeros of $\zeta(s)$.

Let $\chi$ be a Dirichlet character modulo $q$. A generalization of $\zeta(s)$ is the Dirichlet $L$-function $L(s, \chi)$ which is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}.$$ 

Ju.V. Linnik [6], V. G. Sprindzuk [10] and A. Fujii [4], [5] observed that the vertical distribution of the zeros of $\zeta(s)$ is closely related to the horizontal distribution of the zeros.
of $L(s, \chi)$. First, in [6], Linnik has shown the formula

$$
\sum_{n=1}^{\infty} \Lambda(n) \chi(n) e^{-\pi n} = -\frac{1}{\tau(\chi)} \sum_{a=1}^{q} \chi(a) \sum_{\rho: \Re(\rho)>0} \frac{\Gamma(\rho)}{\Gamma(1) \rho} \left( x - 2\pi i \frac{a}{q} \right)^{-\rho} + O(\log^2 x)
$$

(0.1)

where $\rho$ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Sprindzuk extended Linnik’s work, and his paper [10] includes the following theorem.

**Theorem.** Assume that the Riemann Hypothesis is true. Let $\gamma$ run over the imaginary parts of the non-trivial zeros of $\zeta(s)$. Let $q$ be a prime $\geq 3$. Then the generalized Riemann Hypothesis (GRH) for all $L(s, \chi)$ with a character $\chi$ mod $q$ is equivalent to the relation

$$
\sum_{\gamma} |\gamma|^s e^{-s\gamma - \frac{\pi}{4}} \left( x - 2\pi i \frac{a}{q} \right)^{-\frac{1}{2} - i\gamma} = \frac{\mu(q)}{\sqrt{2\pi \varphi(q)}} x^{-1} + O(x^{-\frac{1}{2} - \varepsilon})
$$

(0.2)

as $x \to +0$ for any positive $\varepsilon$ and integer $a$ with $0 < |a| \leq q/2$, $(a, q) = 1$.

Fujii generalized Sprindzuk’s result in [4], [5]. Especially in [5] he obtained the finite sum version of the above theorem of Sprindzuk.

Now we return to Linnik’s formula (0.1). It can be written in the following form

$$
\sum_{n=1}^{\infty} \Lambda(n) \chi(n) e^{-\pi n} = -\sum_{\rho: \Re(\rho) \leq 1} \Gamma(\rho) x^{-\rho} + C_\chi + \frac{1}{2\pi i} \int_{\sigma} L'(s, \chi) \Gamma(s) x^{-s} ds
$$

(0.3)

where $-1 < \sigma < 0$. Together with (0.1) and (0.3), we can see directly a relation between the zeros of $\zeta(s)$ and that of $L(s, \chi)$. In this paper we generalize this type of relation to some $L$-function and its $\chi$-twisted $L$-function. Further we obtain a generalization of Sprindzuk’s theorem in our situation.

The Selberg class $S$, see [7], [9], is a class of Dirichlet series conjecturally containing almost all number theoretically interesting $L$-functions. For example the Dedekind zeta function $\zeta_K(s)$ of an algebraic field $K$, the Hecke $L$-function $L_K(s, \psi)$ associated with a primitive Hecke character $\psi$ and the automorphic $L$-function $L(s, f)$ associated with a holomorphic newform $f(z)$ on some congruence subgroup of $SL_2(\mathbb{Z})$ belong to $S$. Moreover, many other important $L$-functions would belong to $S$, if certain well known conjectures, such as the standard conjecture, or the Langlands conjecture, would hold.