A generalization of Bochner-Weil’s theorem and Stone’s theorem on foundation semigroups

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Abstract. In this paper, we improve a result of Youssfi. As a consequence we obtain an extension for the Bochner-Weil theorem and the Stone theorem on a weighted foundation ∗-semigroup with an identity whose weight is assumed to be bounded on a neighbourhood of the identity.

Introduction. In 1994, Youssfi in his paper [11] gave a positive answer to a problem raised by Choquet in [3] by proving the Bochner-Weil theorem for the $w$-bounded and continuous at the identity positive definite functions on commutative ∗-semigroups with an identity and with a weight function $w$ continuous at the identity. Recently, Ressel and Ricker proved a version of the Stone theorem on commutative discrete semigroups (see, [8]).

In the present paper, we shall first extend Youssfi’s result to the case of topological ∗-semigroups with weight functions which are bounded only on a neighbourhood of the identity. Applications of the obtained results has provided us with a generalization of the Bochner-Weil theorem on foundation ∗-semigroups and topological ∗-groups with weight functions not necessary bounded on a neighbourhood of the identity. We have closed the paper with a generalization of the Ressel and Ricker version of the Stone theorem on foundation ∗-semigroups. It should be noted that the family of foundation ∗-semigroups is very large, for which discrete ∗-semigroups and locally compact ∗-groups are elementary examples. For a wide class of examples of such semigroups we refer the interested reader to the Appendix B of [9].

Preliminaries. Throughout this paper, $S$ will denote a commutative locally compact Hausdorff topological semigroup with an identity $e$. A topological semigroup $S$ is called a ∗-semigroup if there is a continuous mapping $*$ : $S \to S$ such that $(x^*)^* = x^*$ and $(xy)^* = x^*y^*$ for all $x, y \in S$. A function $w : S \to [0, +\infty]$ with $w(e) = 1$ and $w(xy^*) \leq w(x)w(y)$ for all $x, y \in S$ is called a weight function on $S$. A function $\varphi : S \to \mathbb{C}$

is called \textit{w-bounded} if there is a positive constant \( k \) such that \(|\varphi(x)| \leq k w(x)\) for all \( x \in S \). A complex-valued function on \( S \) is called \textit{locally bounded} if it is bounded on each compact subset of \( S \). A \(*\)-semicharacter on \( S \) is any nonzero function \( \chi : S \to \mathbb{C} \) such that \( \chi(xy^n) = \chi(x)\chi(y) \) for all \( x, y \in S \). We denote by \( S^\ast \) the set of all \(*\)-semicharacters on \( S \). Note that when \( S^\ast \) equipped with the topology of pointwise convergence inherited from \( \mathbb{C}^S \), defines a completely regular space. We also note that a \(*\)-semicharacter \( \chi \) is \( w \)-bounded with respect to a weight \( w \) if and only if \( |\chi| \leq w \). Hence \( \Gamma^\ast_w \), the space of all \( w \)-bounded \(*\)-semicharacters on \( S \) is a compact subset of \( S^\ast \). We denote by \( \Gamma^\ast_{w,c,e} \) the (\( \Gamma^\ast_{w,c} \), respectively) the set of all semicharacters in \( \Gamma^\ast_w \) which are continuous at \( e \) (continuous on \( S \), respectively).

Let \( \lambda \) be a nonnegative Radon measure on \( S^\ast \); the generalized Laplace transform of \( \lambda \) whenever it is defined is given by

\[
\tilde{\lambda}(s) = \int_{S^\ast} \gamma(s) d\lambda(\gamma) \quad (s \in S).
\]

These functions are referred to as moment functions (see, [11]). A complex-valued function \( \varphi \) on \( S \) is called \textit{positive-definite} whenever

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c}_j \varphi(x_i x_j^*) \geq 0
\]

for all choices \( \{x_1, \ldots, x_n\} \) of \( S \) and \( \{c_1, \ldots, c_n\} \) of \( \mathbb{C} \). Note that every moment function is positive definite. We denote by \( \mathcal{P}(S, w, c, e) \) the set of all \( w \)-bounded continuous at identity (continuous, respectively) positive-definite functions on \( S \). We denote the complex span of \( \mathcal{P}(S, w, c, e) \) by \( \langle \mathcal{P}(S, w, c, e) \rangle \). As is shown in Proposition 1 of [7] \( \langle \mathcal{P}(S, w, c, e) \rangle \) is translation invariant, that is \( \ell_a \varphi \in \langle \mathcal{P}(S, w, c, e) \rangle \) for every \( \varphi \in \langle \mathcal{P}(S, w, c, e) \rangle \) and \( a \in S \), where \( \ell_a \varphi(x) = \varphi(ax) \) for all \( x \in S \). Let \( w \) be a weight function on \( S \) bounded on a fixed neighbourhood \( V_0 \) of \( e \), and \( V \) be a basis of neighbourhoods \( V \) of \( e \) which are contained in \( V_0 \). For \( V \in \mathcal{V} \) and \( \varphi \in \mathcal{P}(S, w, c, e) \), set

\[
\|\varphi\|_V = \sup\{|\varphi(s)| : s \in V\}.
\]

Let \( \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) denote the complex-vector space of all linear functionals \( L \) on \( \langle \mathcal{P}(S, w, c, e) \rangle \) such that for every \( V \in \mathcal{V} \) there exists a positive number \( C_V \) satisfying

\[
|L(\varphi)| \leq C_V \|\varphi\|_V \quad (\varphi \in \langle \mathcal{P}(S, w, c, e) \rangle, V \in \mathcal{V}).
\]

The infimum of the constants \( C_V \) will be denoted by \( \|L\|_V \). Note that \( \| \|_V \) defines a norm on \( \langle \mathcal{P}(S, w, c, e) \rangle^\ast \). The topology on \( \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) will be the topology induced by the norm \( \| \|_V \), that is a net \( (L_\alpha) \) in \( \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) converges to \( L \in \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) if \( \|L_\alpha - L\|_V \to 0 \) for every \( V \in \mathcal{V} \). A functional \( L \in \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) is called \textit{nonnegative} on \( V \in \mathcal{V} \) if \( L(\varphi) \geq 0 \) for every \( \varphi \in \langle \mathcal{P}(S, w, c, e) \rangle \) with \( \varphi \geq 0 \) on \( V \). A topological \(*\)-semigroup \( S \) is called \textit{admissible with respect to a weight} \( w \) if for each \( V \in \mathcal{V} \), there exists an element \( L = L_V \in \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) which is nonnegative on \( V \) and \( s \to \ell_s L \) from \( S \) into \( \langle \mathcal{P}(S, w, c, e) \rangle^\ast \) is continuous at \( e \), where \( \ell_s L(\varphi) = L(\ell_s(\varphi)) \) for all \( \varphi \in \langle \mathcal{P}(S, w, c, e) \rangle \).