A note on the $p$-local rank

By

BAOSHAN WANG

Abstract. In this paper, we give an inductive definition of $p$-local rank of a $p$-block in a finite group $G$ with $p | |G|$ and show a necessary and sufficient condition for a $p$-block $B$ such that $p lr(B) = 2$.

1. Introduction. Let $G$ be a finite group and $p$ a prime divisor of $|G|$. Given a chain of $p$-subgroups

$$\sigma : Q_0 < Q_1 < \ldots < Q_n$$

of $G$, define the length $|\sigma| = n$, the final subgroup $V^{\sigma} = Q_n$, the initial subgroup $V_{\sigma} = Q_0$, the $k$-th initial sub-chain

$$\sigma_k : Q_0 < Q_1 < \ldots < Q_k,$$

and the normalizer

$$N_G(\sigma) := G_{\sigma} := N_G(Q_0) \cap N_G(Q_1) \cap \ldots \cap N_G(Q_n).$$

Write $\mathcal{C}(G|Q)$ for the set of those chains with initial subgroup $Q$ and $\text{Bl}(G)$ for the set of all $p$-blocks of $G$.

Briefly, we say that a $p$-subgroup $Q$ of $G$ is radical if $Q = O_p(N_G(Q))$, where $O_p(H)$ is the unique maximal normal $p$-subgroup of $H$. We say that the $p$-chain $\sigma$ is radical if $Q_i = O_p(N_G(\sigma_i))$ for each $i$, i.e., if $Q_0$ is a radical $p$-subgroup of $G$ and $Q_i$ is a radical $p$-subgroup of $N_G(\sigma_{i-1})$ for each $i \neq 0$. Write $\mathcal{R} = \mathcal{R}(G)$ for the set of radical $p$-chains of $G$ and write $\mathcal{R}(G|Q) = \{\sigma \in \mathcal{R}(G) : V_{\sigma} = Q\}$. Write $\mathcal{R}(G|Q)/G$ for a set of orbit representatives under the action of $G$. Following [6], the $p$-local rank $\text{plr}(G)$ of $G$ is the length of a longest chain in $\mathcal{R}(G)$. We say that a subgroup $H$ of $G$ is a trivial intersection (T. I.), if $H^g \cap H = 1$ for every $g \in G \setminus N_G(H)$. By [6, 7.1], if $\text{plr}(G) > 0$, then $\text{plr}(G) = 1$ if and only if $G/O_p(G)$ has T. I. Sylow $p$-subgroups.

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The $p$-local rank is first defined in [6]. For convenience, we restate the original definition below.

Let $G$ be a finite group with $O_p(G) = 1$. Let $P$ be a subgroup of $G$ with $O_p(P) \neq 1$. We say that $P$ is a level 1 $p$-parabolic subgroup of $G$ if $P = N_G(O_p(P))$. We adjoin other parabolic subgroups of other levels as follows: if $Q$ is a parabolic subgroup of level $i$ and $\overline{X}$ is a level 1 $p$-parabolic subgroup of $Q = Q/O_p(Q)$, we call $X$, the full pre-image of $\overline{X}$ in $Q$, a level $(i + 1)$ $p$-parabolic subgroup of $G$.

We define the $p$-local rank ($p\text{lr}$) of finite group $H$ inductively as follows:

1. if $H$ is a $p'$-group, then $p\text{lr}(H) = 0$;
2. if $O_p(H) \neq 1$, then $p\text{lr}(H)$ is by definition equal to $p\text{lr}(H/O_p(H))$;
3. if $O_p(H) = 1$, but $H$ is not a $p'$-group, then $p\text{lr}(H)$ is by definition equal to $1 + \max\{p\text{lr}(P) \mid P$ is a level 1-parabolic subgroup of $H\}$.

Now, let $(K, R, k)$ be a $p$-modular system which splits for $G$, and consider a $p$-block $B$ of $G$ with defect group $\delta(B)$. Given a subgroup $H$ of $G$, write $\text{Bl}(H|B) = \{b \in \text{Bl}(H) \mid b^G = B\}$ (in the sense of Brauer).

The Knörr-Robinson reformulation of Alperin’s weight conjecture (see [5] and [1]) states that if $B$ has positive defect then $\sum_{\sigma \in \mathcal{C}(G|1)} (-1)^{|\sigma|}l(G_\sigma, B) = 0$, where $l(G_\sigma, B)$ denotes the number of simple $kG_\sigma$-modules belonging to $p$-blocks of $G_\sigma$ whose Brauer correspondent is $B$. It is clear that if $l(G_\sigma, B) \neq 0$ then there is some $p$-block $b$ of $G_\sigma$ with $b^G = B$.

It thus makes sense to consider the set

$$\mathcal{C}(G, B) = \{\sigma \in \mathcal{C}(G) \mid \text{Bl}(G_\sigma|B) \neq \emptyset\}.$$ 

We say that $\sigma$ is a $p$-chain for $B$, or a $B$-chain (as we will see we may associate with $\sigma$ a chain of $B$-subgroups in the sense of Alperin [2]) if $\sigma \in \mathcal{C}(G, B)$.

We write

$$\mathcal{R}(G, B) = \{\sigma \in \mathcal{R}(G) \mid \text{Bl}(N_G(\sigma)|B) \neq \emptyset\}.$$ 

As in [3], we define the $p$-local rank $p\text{lr}(B)$ of $B$ to be the number

$$p\text{lr}(B) = \max\{|\sigma| \mid \sigma \in \mathcal{R}(G, B)\}.$$ 

Clearly, $p\text{lr}(G) \geq p\text{lr}(B)$ for all $B \in \text{Bl}(G)$. As shown in [3], we have that for the principal block $B_0$ of $G$, $p\text{lr}(B_0) = p\text{lr}(G)$ follows Brauer’s Third Main Theorem.

The paper is organized as follows. In Section 2 we consider our definitions in terms of subpairs. In Section 3, we give some notations and some lemmas. In the last section, we will prove our main results.

2. **Subpairs.** A pair $(Q, b)$ is called a *Brauer pair*, if $Q$ is a $p$-subgroup of $G$ and $b$ a block of $QC_G(Q)$ with defect group $Q$. Generally, if $Q$ is a $p$-subgroup of $G$ and $b_Q$ is a block of $QC_G(Q)$ then the pair $(Q, b_Q)$ is called a *subpair* (for “subgroup block pair”) of $G$. There we make no restriction on the defect groups of $b_Q$. Each subpair corresponds