A Ricci inequality for hypersurfaces in the sphere

By

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Abstract. Let \( M^n \) be a complete Riemannian manifold immersed isometrically in the unity Euclidean sphere \( S^{n+1} \). In [9], B. Smyth proved that if \( M^n, n \geq 3 \), has sectional curvature \( K \) and Ricci curvature \( \text{Ric} \), with \( \inf K > -\infty \), then \( \sup \text{Ric} \geq (n-2) \) unless the universal covering \( \tilde{M}^n \) of \( M^n \) is homeomorphic to \( \mathbb{R}^n \) or homeomorphic to an odd-dimensional sphere. In this paper, we improve the result of Smyth. Moreover, we obtain the classification of complete hypersurfaces of \( S^{n+1} \) with nonnegative sectional curvature.

Introduction. Let \( M^n \) be a Riemannian \( n \)-manifold and \( \tilde{M}^n \) the universal covering of \( M^n \) endowed with the metric of the covering. Let \( S^k_c \) be the \( k \)-dimensional sphere with sectional curvature \( c \), \( S^{n+1} \) the unity Euclidean \( (n+1) \)-sphere and \( \mathbb{R}^n \) the Euclidean \( n \)-space. Let \( K \) be the sectional curvature of \( M^n \) and \( \text{Ric} \) its Ricci curvature. In [5], Efimov proved that for a complete surface \( M^2 \) of \( \mathbb{R}^3 \), we have that \( \sup K \geq 0 \). This result generalizes the Hilbert theorem which asserts that the hyperbolic plane cannot be immersed isometrically in \( \mathbb{R}^3 \). The theorem of Efimov was partially generalized by B. Smyth and F. Xavier [11] for hypersurfaces of \( \mathbb{R}^{n+1} \). The corrected form of Efimov’s inequality for a hypersurface of \( S^{n+1} \) was obtained by Smyth in [9]:

**Theorem** [B. Smyth]. Let \( f: M^n \longrightarrow S^{n+1}, n \geq 3 \), be a complete oriented hypersurface with Ricci curvature \( \text{Ric} \) and with sectional curvature \( K \) such that \( \sup K > -\infty \). Then we have only two possibilities (1) or (2):

1) \( \sup \text{Ric} \geq n - 2 \).

2) \( \sup \text{Ric} < n - 2 \) and in this case we have:
   a) If \( M^n \) is compact, then \( n \) is odd and \( \tilde{M}^n \) of \( M^n \) is homeomorphic to \( S^n \).
   b) If \( M^n \) is non compact, then \( M^n \) is homeomorphic to \( \mathbb{R}^n \).

Also, in [10, Theorem 2], B. Smyth proved that if \( M^n \) is compact oriented and \( f: M^n \longrightarrow S^{n+1} \) is an isometric immersion with constant mean curvature, then \( \sup \text{Ric} \geq n - 2 \). In [8, Theorem A], Hasanis and Vlachos proved the following:

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Theorem [Hasanis-Vlachos]. Let $f : M^n \to S^{n+1}$, $n \geq 3$, be a complete oriented and minimal hypersurface. Then $\sup \text{Ric} \geq n - 2$. Moreover, we have

1) If $n$ is even, then $\sup \text{Ric} = n - 2$ if and only if $f(M^n)$ is isometric to the Clifford torus $S^{n/2}_2 \times S^{n/2}_2$.
2) If $n$ is odd and $\sup \text{Ric} = n - 2$, then $\tilde{M}^n$ is homeomorphic to $S^n$.

In this paper, we improve the theorem of Smyth, see Theorem A, the one of Hasanis/Vlachos, see Corollary A:

**Theorem A.** Let $f : M^n \to S^{n+1}$, $n \geq 3$, be a complete and oriented hypersurface with mean curvature $H$. Then:

1) If $H$ is constant, then $\sup \text{Ric} \geq n - 2$.
2) If $H$ is bounded and $\sup \text{Ric} \leq n - 2$, then there are two possibilities:
   a) If $M^n$ is compact then $\hat{M}^n$ is homeomorphic to $S^n$ and $n$ is odd or $n = 2$ and $\hat{M}^2$ is homeomorphic to $\mathbb{R}^2$, or $n > 3$ and $f(M^n)$ is isometric to the Clifford torus $S^k_{c_1} \times S^{n-k}_{c_2}$. $H$ is constant and $\text{Ric} \equiv n - 2$. Moreover, in the last case, we have that $c_1 = \frac{n-2}{k-1}, c_2 = \frac{n-2}{n-k-1}, k > 1, n-k > 1$ and
   \[
   |H| = \frac{|n-2k|}{\sqrt{n(n-1)(n-k-1)}}.
   \]
   b) If $M^n$ is non compact and $n \geq 3$, then $\tilde{M}^n$ is homeomorphic to $\mathbb{R}^n$.
3) If $n \geq 3$, $H$ is unbounded, $\sup \text{Ric} \leq n - 2$ and $\inf K > -\infty$, where $K$ denotes the sectional curvature of $M^n$, then $M^n$ is homeomorphic to $\mathbb{R}^n$.

**Corollary A.** Let $f : M^n \to S^{n+1}$, $n \geq 2$, be a complete oriented hypersurface with constant mean curvature $H$. Then $\sup \text{Ric} \geq n - 2$. Moreover,

1) If $n = 2$ and $\sup \text{Ric} = 0$, then $\hat{M}^2$ is homeomorphic to $\mathbb{R}^2$.
2) If $n > 2$ and $\sup \text{Ric} \leq n - 2$, then $M^n$ is homeomorphic to $S^n$ and $n$ is odd or $f(M^n)$ is isometric to the Clifford torus $S^k_{c_1} \times S^{n-k}_{c_2}$, $\text{Ric} \equiv n - 2$, where $k > 1, n-k > 1$, $c_1 = \frac{n-2}{k-1}, c_2 = \frac{n-2}{n-k-1}$ and
   \[
   |H| = \frac{|n-2k|}{\sqrt{n(n-1)(n-k-1)}}.
   \]

Remark. We don’t know an example showing that the alternative $\sup \text{Ric} = n - 2$ and $\text{Ric} \neq n - 2$ can actually occur in Theorem A and Corollary B.

In the proof of the Theorem A, we use the following result.

**Theorem B.** Let $f : M^n \to S^{n+1}_c$, $n \geq 3$, be an isometric immersion, where $M^n$ is a complete oriented Riemannian $n$-manifold. Suppose that $M^n$ has sectional curvature $K \geq 0$. Then: