Polarizations of Prym varieties of pairs of coverings

By

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Abstract. To any pair of coverings \( f_i : X \to X_i, i = 1, 2 \), of smooth projective curves one can associate an abelian subvariety of the Jacobian \( J_X \), the Prym variety \( P(f_1, f_2) \) of the pair \( (f_1, f_2) \). In some cases we can compute the type of the restriction of the canonical principal polarization of \( J_X \). We obtain 2 families of Prym-Tyurin varieties of exponent 6.

1. Introduction. Let \( f : X \to Y \) be a finite morphism of smooth projective curves. The complement of the abelian subvariety \( f^* J_Y \) in the canonically polarized Jacobian \( J_X \) of \( X \) is called the Prym variety \( P(f) \) of \( f \). In [3] we introduced an analogous notion for two morphisms of curves. To be more precise, suppose that we are given a commutative diagram of finite morphisms of smooth projective curves:

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & X_1 \\
\downarrow{f_2} & & \downarrow{g_2} \\
X_2 & \xleftarrow{g_1} & Y \\
\end{array}
\]

(1.1)

such that \( g_1 \) and \( g_2 \) do not both factorize via the same morphism \( Y' \to Y \) of degree \( \geq 2 \). Then the Prym variety \( P(f_1, f_2) \) of the pair \( (f_1, f_2) \) is defined to be the complement of the abelian variety \( f_2^* (P(g_2)) \) in \( P(f_1) \) with respect to the canonical polarization.

By definition \( P(f_1, f_2) \) is an abelian subvariety of the Jacobian \( J_X \) of the curve \( X \). In general it is difficult to determine the type of the restriction of a polarization to an abelian subvariety. It is the aim of this note to determine the type of the restriction of the canonical

†Sevin Recillas died on June 20, 2005.
polarization of $J_X$ to $P(f_1, f_2)$ in some cases, namely for $Y = \mathbb{P}^1$ and deg $f_1$ and deg $f_2$ are prime to each other. As a special case we obtain 2 families of Prym-Tyurin varieties of exponent 6 in any dimension $\geq 4$ respectively 5. Note that the correspondences defining $P(f_1, f_2)$ are not fixed point free. So one cannot apply Kanev’s theorem for the proof.

In the last section we show that the Abel-Prym map of these abelian varieties is injective apart from the fact that all ramification points of $f_1$ are mapped to one point. Hence we obtain irreducible curves of low class in $P(f_1, f_2)$. Moreover this implies that the Seshadri constant of $P(f_1, f_2)$ is small in these cases.

2. Restricting polarizations. Let $(S, L)$ be a polarized abelian variety over an algebraically closed field of characteristic 0. As usual let $K(L)$ denote the kernel of the associated homomorphism $\phi_L : S \to \hat{S}, x \mapsto t^*_x L \otimes L^{-1}$. The polarization $L$ is of type $(d_1, \ldots, d_g)$ if and only if $K(L) \simeq (\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_g\mathbb{Z})^2$.

Now let $A$ be an abelian subvariety of $S$ with canonical embedding $\iota_A : A \to S$. Let
\begin{equation}
P := \ker(\iota_A \circ \phi_L : S \to \hat{A})^0
\end{equation}
denote the complementary abelian subvariety of $S$. Let $L_A := L|A$ and $L_P := L|P$ denote the restriction of the polarization $L$ to $A$ and $P$. If $L$ is a principal polarization, it is well known that
\begin{equation}
K(L_A) \simeq K(L_P) \simeq A \cap P
\end{equation}
from which the type of $L_A$ and $L_P$ can be deduced in many cases. For arbitrary polarizations the situation is more complicated. In this section we recall two generalizations of (2.2) which will be applied later and for the proof of which we refer to [2].

Proposition 2.1. $|K(L_P)| \cdot |K(L_A)| = |A \cap P|^2 \cdot |K(L)|$.

Proposition 2.2. There is an exact sequence
\[ 0 \to K(L) \cap P \to K(L_P) \to A \cap P \to 0. \]
In particular $|K(L_P)| = |K(L) \cap P| \cdot |A \cap P|$ and similarly for $L_A$.

3. Restriction of the canonical polarization to $P(f_1, f_2)$. Let $f : X \to Y$ be a morphism of degree $n$ of smooth projective curves over an algebraically closed field $k$. Denote by $J_X := \text{Pic}^0(X)$ and $J_Y := \text{Pic}^0(Y)$ the Jacobians of $X$ and $Y$. Pulling back line bundles defines a homomorphism
\[ f^* : J_Y \to J_X. \]
$f^*$ has finite kernel and is an embedding if and only if $f$ does not factor via a cyclic étale cover of degree $\geq 2$ (see [1, Proposition 11.4.3]). The norm map of line bundles defines a homomorphism
\[ N_f : J_X \to J_Y. \]