Unbounded solutions of differential equations of second order

By

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Abstract. In this paper, the existence of unbounded solutions for the following nonlinear asymmetric oscillator

\[ x'' + f(x)x' + \alpha x^+ - \beta x^- = h(t) \]

is discussed, where \( \alpha, \beta \) are positive constants satisfying

\[ \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{\omega} \]

for some \( \omega \in \mathbb{R}^+/\mathbb{Q} \), \( h(t) \in L^\infty[0, 2\pi] \) is \( 2\pi \)-periodic, \( x^\pm = \max(\pm x, 0) \).

1. Introduction. We consider in this paper the existence of unbounded solutions for the following asymmetric oscillator

(1) \[ x'' + f(x)x' + \alpha x^+ - \beta x^- = h(t), \quad (\equiv d/dt) \]

where \( x^\pm = \max(\pm x, 0) \), \( \alpha, \beta \) are positive constants satisfying

(2) \[ \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{\omega} \]

for some \( \omega \in \mathbb{R}^+/\mathbb{Q} \), \( h \in L^\infty[0, 2\pi] \) is \( 2\pi \)-periodic. If \( f(x) \equiv 0, \alpha \neq \beta \) and there exist \( n, m \in \mathbb{N} \) such that

(3) \[ \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2m}{n} \]

then the unboundedness problem of solutions of (1) was discussed by Alonso and Ortega [1].

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For $\alpha > 0, \beta > 0$ satisfying (3), let $C(t)$ be the solution of the following initial value problem

$$
\begin{align*}
&x'' + \alpha x^+ - \beta x^- = 0, \\
&x(0) = 1, \quad x'(0) = 0.
\end{align*}
$$

(4)

Then it is well-known that $C(t) \in C^2(R)$ is $\frac{2m\pi}{n}$-periodic. Define a $\frac{2m\pi}{n}$–periodic function

$$
\Phi_h(\theta) = \frac{2\pi}{n} \int_0^{2\pi} C\left(\frac{m\theta}{n} + t\right) h(t) dt, \quad \theta \in \mathbb{R}.
$$

Then it is proved in [1] that if the set

$$
\Omega = \{\theta \in S^1, \Phi_h(\theta) = 0\}
$$

is nonempty and for every $\theta \in \Omega, \Phi'_h(\theta) \neq 0$, then there exists $R_0 > 0$ such that every solution $x(t)$ of (1) with initial value $(x(t_0), x'(t_0))$ such that

$$
x^2(t_0) + (x'(t_0))^2 > R_0^2
$$

for some $t_0 \in R$, goes to infinity in the future or in the past.

Recently, Wang [8] obtained sufficient conditions for the unboundedness of solutions of equation (1) where $\alpha, \beta$ satisfy (2). By using the well-known Birkhoff Ergodic Theorem, it is proved in [8] that if the limits

$$
\lim_{x \to \pm \infty} \int_0^x f(s) ds =: F(\pm \infty)
$$

exist and are bounded. Then all solutions of (1) with large initial values are unbounded provided $F(+\infty)F(-\infty) < 0$.

For more recent results on the existence of periodic or unbounded solutions of second order differential equations which is similar to (1) if $\alpha$ and $\beta$ satisfy (3), we refer to [2–6] and the references therein. In this paper, inspired by the work of Wang, we obtain some sufficient conditions for the existence of unbounded solutions for equation (1) if some conditions in [8] are not satisfied, the results obtained in this paper are natural refinements of the work of [8].

The main results of this paper are

**Theorem 1.** Let $S(t)$ be the solution of the following initial value problem

$$
\begin{align*}
&u'' + \alpha u^+ - \beta u^- = 0, \\
&u(0) = 0, \quad u'(0) = 1,
\end{align*}
$$

(5)

where $\alpha, \beta$ satisfy (2).