On toric \(h\)-vectors of centrally symmetric polytopes

By

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Abstract. We prove tight lower bounds for the coefficients of the toric \(h\)-vector of an arbitrary centrally symmetric polytope generalizing previous results due to R. Stanley and the author using toric varieties. Our proof here is based on the theory of combinatorial intersection cohomology for normal fans of polytopes developed by G. Barthel, J.-P. Brasselet, K. Fieseler and L. Kaup, and independently by P. Bressler and V. Lunts. This theory is also valid for nonrational polytopes when there is no standard correspondence with toric varieties. In this way we can establish our bounds for centrally symmetric polytopes even without requiring them to be rational.

Introduction. The \(h\)-vector of a simplicial polytope is defined as a linear combination of the face numbers of the polytope, and its entries have a topological interpretation as the Betti numbers of the quasi-smooth toric variety corresponding to the polytope. For the \(h\)-vector of a centrally symmetric simplicial polytope, R. Stanley proved tight lower bounds (see [10]) using the theory of toric varieties. His results were then generalized by R. Adin who considered simplicial polytopes with a certain symmetry of prime power order (see [3]).

For arbitrary polytopes, R. Stanley introduced the notion of a generalized \(h\)-vector (see [11]). This combinatorial invariant of the polytope is defined by recursion over its faces, and in the simplicial case it coincides with the usual \(h\)-vector. If the polytope \(P\) is rational then there is an associated projective toric variety \(X_P\), and the coefficients of the generalized \(h\)-vector of \(P\) are the Betti numbers of the intersection cohomology of middle perversity of \(X_P\). Because of this interpretation, the generalized \(h\)-vector is also referred to as the toric \(h\)-vector of the polytope (see [6]).

In a previous article, we considered rational polytopes with the type of symmetry investigated by Adin (see [1]). We asked for conditions imposed by the existence of the symmetry on the toric \(h\)-vector of the polytope. Using its topological interpretation via intersection cohomology of middle perversity, we obtained tight lower bounds for the coefficients of the toric \(h\)-vector in this situation. In the case of a centrally symmetric simplicial polytope, we get back the bounds due to Stanley (see [10]).

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The aim of this article is to show that the same bounds remain valid even if we do not assume the centrally symmetric polytope to be rational. Our proof is based on the theory of combinatorial intersection cohomology for fans developed by G. Barthel, J.-P. Brasselet, K. Fieseler and L. Kaup (see [5]) and independently by P. Bressler and V. Lunts (see [7]).

They discovered that one can completely characterize the intersection cohomology of middle perversity of a toric variety by combinatorial and algebraic data associated to the corresponding fan, namely in terms of a minimal extension sheaf on the fan considered as a topological space where the subfans are the open subsets (see [4]). Associating an analogous object to a non-rational fan, they define a combinatorial intersection cohomology satisfying similar formal properties as the usual intersection cohomology.

Both teams of authors conjectured that for the combinatorial intersection cohomology of a fan arising from a polytope, a combinatorial version of the Hard Lefschetz Theorem holds. Moreover, they proved that if such a Hard Lefschetz theorem is true then the even Betti numbers of the combinatorial intersection cohomology are precisely the coefficients of the toric $h$-vector of the corresponding polytope (see [5] and [7]).

The Hard Lefschetz Theorem in this context was recently proved by Kalle Karu (see [9]). The fact that a combinatorial Hard Lefschetz Theorem holds has striking consequences. For example, the coefficients of the toric $h$-vector of an arbitrary polytope are non-negative which is not at all clear from the definition. Moreover, the toric $h$-vector of an arbitrary polytope is unimodal.

We apply these results to a centrally symmetric polytope $P$ of dimension $n$. Denoting its toric $h$-vector by $(h_0, \ldots, h_n)$, we prove the following for the polynomial $h_P := \sum_{j=0}^n h_j x^j$ (see Theorem 4.2):

**Theorem.** If a polytope $P$ of dimension $n$ admits a central symmetry then the polynomial

$$h_P(x) - (1 + x)^n$$

has nonnegative, even coefficients, it is palindromic and unimodal. That means that we have the following bounds for the coefficients $h_j$ of $h_P$:

$$h_j - h_{j-1} \geq \binom{n}{j} - \binom{n}{j-1} \quad \text{for } j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Note that $(1 + x)^n$ occurs as the $h$-polynomial of the $n$-dimensional cross-polytope. We can reformulate the lower bounds given by the theorem in terms of the partial ordering on real polynomials of degree $n$ defined by coefficientwise comparison, i.e., $a = \sum_{j=0}^n a_j x^j \leq b = \sum_{j=0}^n b_j x^j$ if and only if $a_j \leq b_j$ for all $j$. The $h$-polynomial of the $n$-dimensional cross-polytope is minimal in this sense and in fact this is the only polytope realizing the minimum (see Corollary 4.3).