Confined Banach spaces

By

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Abstract. We discuss smoothness of the Weyl functional calculus and use it to prove that every C*-algebra is a confined Banach space.

1. Introduction. A Banach space \( X \) is said to be confined if there exists a bounded \( C^\infty \) mapping \( f : X \rightarrow X \) such that \( f(x) = x \) for all \( x \) on some neighbourhood of the origin. Confined Banach spaces arise in the work of C. J. Atkin who considers the problem of replacing the assumption of the existence of a smooth partition of unity (see [5]) with weaker and more accessible conditions (see [2], [3]). More recently, L. Lempert [6] has used confined spaces to extend the collection of Banach spaces which admit a positive solution to the Cousin problem. Atkin gives a number of examples of confined spaces, such as \( C(K) \). The main result in this paper is to show that any C*-algebra is a confined space. To obtain this result we use the non-commutative Weyl functional calculus, originally defined in [8], and which has since been extended to arbitrary self-adjoint operators by Anderson [1] and Taylor [7]. In Section 2 of this paper we discuss \( C^\infty \) mappings between Banach spaces, in Section 3 we discuss the Weyl functional calculus and in the final section we show that C*-algebras are confined Banach spaces.

2. Smooth mappings between Banach spaces. An \( n \)-homogeneous polynomial between Banach spaces \( X \) and \( Y \) is the restriction to the diagonal of an \( n \)-linear mapping from \( X \times \ldots \times X \) to \( Y \). If \( f : U \subset X \rightarrow Y \) (\( U \) open) and \( x \in U \) we say that \( f \) is (Fréchet) differentiable at \( x \) if there exists \( T \in L(X, Y) \), the space of bounded linear operators from \( X \) to \( Y \) endowed with the operator norm, such that

\[
\lim_{y \to 0} \frac{\| f(x + y) - f(x) - T(y) \|}{\| y \|} = 0.
\]


This work was carried out with the partial support of Science Foundation Ireland grant R9317.
The mapping $T$ is necessarily unique and is called the derivative of $f$ at $x$, written $f'(x)$ or $df(x)$. If $f$ is differentiable at all points of $U$ and the mapping $df : x \in U \rightarrow df(x) \in L(X, Y)$ is continuous then we say that $f \in \mathcal{C}^1(U, Y)$ and call $f$ a continuously differentiable function. By induction, we let $d^m f := d(d^{m-1} f)$ for $m > 1$ and define $\mathcal{C}^m(U, Y)$ as the set of all $f$ such that $d^m f$ exists and is continuous. Since differentiable mappings are continuous it follows that $f \in \mathcal{C}^\infty(U, Y)$ if and only if $f$ has an $m$th order derivative at each point $x$ in $U$ for all $m$.

By the well-known correspondence between $n$-homogeneous polynomials and symmetric $n$-linear mappings and the fact that the order of partial differentiation for smooth functions can be interchanged, we see that $f : U \subset X \rightarrow Y \in \mathcal{C}^\infty(U, Y)$ if and only if for all $x \in U$ there exists a sequence $(P_{j,n})_{j=0}^n$, $P_{j,n} \in \mathcal{C}^\infty(U, Y)$ (the continuous $f$-homogeneous polynomials form $X$ to $Y$) such that for some $\delta > 0$ we have

$$\lim_{\epsilon \to 0} \sup_{\|y\| = \delta} \left\| f(x + \epsilon y) - \sum_{j=0}^n \epsilon^j P_{j,n}(y) \right\| = 0.$$  \hspace{1cm} (1)

Uniqueness of Taylor series expansions implies that $P_{j,n+1} = P_{j,n}$ for all $n$ and all $j \leq n$.

3. The Weyl functional calculus. Let $\mathcal{S}[\mathbb{R}]$ denote the self-adjoint elements in the unital $\mathbb{C}^*$-algebra $\mathcal{A}$. The Weyl functional calculus was first defined for finite self-adjoint tuples by M.E. Taylor [7] and afterwards extended by R.F.V. Anderson [1]. Various collections of function spaces can be considered and we have just confined ourselves to the collection suitable for our purposes. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing real-valued functions on $\mathbb{R}^n$ with the topology of uniform convergence of the function and its partial derivatives. Let $\mathcal{E}(\mathbb{R}^n)$ denote the Fréchet space of real-valued $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ endowed with the topology of uniform convergence of the function and its partial derivatives on compact subsets of $\mathbb{R}^n$. Let $\mathcal{F}$ denote the Fourier transform and for $A := (A_1, \ldots, A_n) \in \mathcal{N}[\mathbb{R}]$ and $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ we let $\langle \lambda, A \rangle = \sum_{i=1}^n \lambda_i A_i$.

The mapping $\xi : (f, A) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{N}[\mathbb{R}] \mapsto f(A) \in \mathcal{S}[\mathbb{R}]$ given by

$$f(A) := \langle f, \mathcal{F}^{-1}(\exp(i\langle \cdot, A \rangle)) \rangle$$

$$= : (\mathcal{F}(f), \exp(i\langle \cdot, A \rangle))$$

$$= : (2\pi)^{-n/2} \int_{\mathbb{R}^n} (\mathcal{F}f)(\lambda) \exp(-i\langle \lambda, A \rangle) d\lambda$$

is well defined and, by [1, Theorem 2.9(c)], we have

$$\|\xi(f, A)\| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|\exp(-i\langle \lambda, A \rangle)\| \|\mathcal{F}(f)\| d\lambda$$

$$= \|\mathcal{F}(\phi f)\|_{L^1}$$