A note on semigroups, groups and geometric lattices

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Abstract. Let $G$ be a closed, additive semigroup in a Hausdorff topological vector space. Then $G$ is a group if and only if it satisfies natural convexity conditions of algebraic or geometric-topological type. This yields a characterization of the geometric lattices among the discrete, additive semigroups of Euclidean $d$-space $\mathbb{E}^d$ and, more generally, of direct sums of subspaces and lattices in $\mathbb{E}^d$.

Mathematics Subject Classification (2000). 11H06; 11H99; 22A15; 39B05; 74E15; 82D25; 90C10.

Keywords. Semigroup, Convexity condition, Topological vector space, Discrete semigroup, Group, Geometric lattice.

1. Introduction and statement of result. In several articles on functional equations, the following result is used: let $a, b \in \mathbb{R}$ be such that $a/b$ is irrational and negative. Then the set $F$ of all numbers of the form $ma + nb$, where $m, n$ are arbitrary non-negative integers, is dense in $\mathbb{R}$. See Matkowski [9], Krassowska, Małolepszy, and Matkowski [8], and Schwaiger [11]. The essential assumption in this result is the non-negativity, or, in other words, that $F$ is a semigroup. Omitting it, we get a simple property of irrational numbers. This led us to the question to characterize groups among vector semigroups.

In this note, we show that a vector semigroup is a group if it satisfies certain natural convexity conditions. The above result then is a simple consequence.

Let $\mathbb{E}$ be a real Hausdorff topological vector space. An additive semigroup in $\mathbb{E}$ is a subsemigroup of the additive group of $\mathbb{E}$. This note deals with the question, when is an additive semigroup $G$ in $\mathbb{E}$ a group? If $G$ is a group then the origin $o$ is contained ‘deep’ in the convex hull of $G$. It turns out that, cum grano salis, this condition is necessary and sufficient for a closed semigroup to be a group. More precisely, we have the following results.
Theorem. Let $\mathbb{E}$ be a real Hausdorff topological vector space and $G$ a closed additive semigroup in $\mathbb{E}$. Then the following statements are pairwise equivalent:

1. $G$ is a group.
2. $\text{conv } G = \text{lin } G$.
3. $o \in \text{relint } \text{conv } G$.
4. $o \in \text{algrelint } \text{conv } G$.

Here, conv, lin and int, relint and algrelint stand for convex and linear hull and for interior, interior relative to the linear hull (which for semigroups is the same as the affine hull), and algebraic relative interior. The algebraic relative interior of a set $A \subseteq \mathbb{E}$ is the set of all $p \in A$ such that every line through $p$ either meets $A$ only at $p$ or in a line segment with $p$ in its relative interior.

The proof of the theorem makes use of a version of Kronecker’s approximation theorem.

The theorem implies that the additive semigroup $F$, which was considered earlier, is dense in $\mathbb{E}$: The semigroup $F$ is dense in its closure $G$. The closed additive semigroup $G$ in $\mathbb{R}$ contains both positive and negative numbers and thus, by the theorem, is a closed additive group in $\mathbb{R}$. It is well known that the closed additive groups in $\mathbb{R}$ are of the form $\alpha \mathbb{Z}$, where $\alpha \in \mathbb{R}$, or coincide with $\mathbb{R}$. Since $a, b \in F \subseteq G$, $a/b$ irrational, the case $G = \alpha \mathbb{Z}$ is excluded and it follows that $F$ is dense in $G = \mathbb{R}$.

The following is an immediate consequence of the theorem: If $\text{int } \text{conv } G \neq \emptyset$, then $G$ is a group if and only if $o \in \text{int } \text{conv } G$.

A (geometric) lattice $L$ of rank $r$ in $\mathbb{E}^d$ is the set of all integer linear combinations of $r$ linearly independent vectors $b_1, \ldots, b_r$ in $\mathbb{E}^d$. The $r$-tuple $\{b_1, \ldots, b_r\}$ is a basis of $L$ and the $r$-dimensional volume of the fundamental parallelootope $\{\lambda_1 b_1 + \cdots + \lambda_r b_r : 0 \leq \lambda_i < 1\}$ is the determinant $d(L)$ of $L$. Lattices, in general of rank $d$, are basic tools of the geometry of numbers and crystallography and play a role in other areas. See, e.g., [3,5,6].

Corollary. Let $G$ be a closed, additive semigroup in $\mathbb{E}^d$. Then the following statements hold:

1. The semigroup $G$ is a group if and only if $o \in \text{algrelint } \text{conv } G = \text{relint } \text{conv } G$.
2. The semigroup $G$ is a group if and only if $G = L \oplus S$, where $L$ is a lattice of rank $r$ and $S$ a linear subspace of dimension $s$, such that $0 \leq r, s$ and $r + s \leq d$.

The second statement is well-known, see Bourbaki, Chapter 7 [1], Siegel, p. 56 [12], and Hewitt and Ross, p. 92 [7]. It is given here to make the picture more complete. The proof is new. It is based on a density argument.

2. Proof of the Theorem. Assume that

1. $G$ is a group.

This implies the following equality,