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An infinite family of Gromoll–Meyer spheres

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Abstract. We construct a new infinite family of models of exotic 7-spheres. These models are direct generalizations of the Gromoll–Meyer sphere. From their symmetries, geodesics and submanifolds half of them are closer to the standard 7-sphere than any other known model for an exotic 7-sphere.

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1. Introduction. This paper provides a new geometric way to construct all exotic 7-spheres. Exotic spheres are differentiable manifolds that are homeomorphic but not diffeomorphic to standard spheres. The first examples were found by Milnor [15] in 1956 among the $\mathbb{S}^3$-bundles over $\mathbb{S}^4$. It turned out that 7 is the smallest dimension where exotic spheres can occur except possibly in the special dimension 4. In any dimension $n > 4$ the exotic spheres and the standard sphere form a finite abelian group: the group Θ_n of (orientation preserving diffeomorphism classes of) homotopy spheres [14]. The inverse element in Θ_n can be obtained by a change of orientation. In dimension 7 we have $\Theta_7 \approx \mathbb{Z}_{28}$. Hence, ignoring orientation there are 14 exotic 7-spheres. From these 14 exotic 7-spheres four (corresponding to $2, 5, 9, 12, 16, 19, 23, 26 \in \mathbb{Z}_{28}$) are not diffeomorphic to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ [7].

In 1974 Gromoll and Meyer [8] constructed an exotic 7-sphere, Σ^7_{GM}, as quotient of the compact group $\mathrm{Sp}(2)$ by a two-sided $\mathbb{S}^3$-action. This construction provided Σ^7_{GM} automatically with a metric of non-negative sectional curvature ($K \geq 0$). The Gromoll–Meyer sphere Σ^7_{GM} was the only exotic sphere known

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to admit such a metric until 1999 when Grove and Ziller [9] constructed metrics with $K \geq 0$ on all Milnor spheres, i.e., on all exotic 7-spheres that are $\mathbb{S}^3$-bundles over $\mathbb{S}^4$. In 2002 Totaro [19] and independently Kapovitch and Ziller [13] showed that Σ^7_{GM} is the only exotic sphere that can be modeled by a biquotient of a compact group and thus underlined the singular status of the Gromoll–Meyer sphere among all models for exotic spheres.

Nevertheless, we provide an elementary and direct generalization of the Gromoll–Meyer construction. The essential components in this construction are natural self-maps of $\mathbb{S}^7$, namely, the n-powers of unit octonions, $n \in \mathbb{Z}$. In terms of quaternions these maps are defined by

$$\rho_n : \mathbb{S}^7 \to \mathbb{S}^7, \quad \left(\begin{smallmatrix} \cos t + p \sin t \\ w \sin t \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \cos nt + p \sin nt \\ w \sin nt \end{smallmatrix}\right)$$

where $p \in \operatorname{Im} \mathbb{H}$ and $w \in \mathbb{H}$ with $|p|^2 + |w|^2 = 1$. Let $\langle\langle u, v \rangle\rangle := \bar{u}^{\mathrm{t}} v$ denote the standard Hermitian product on $\mathbb{H}^2$. The submanifolds

$$E^{10}_n := \left\{(u, v) \in \mathbb{S}^7 \times \mathbb{S}^7 \middle| \langle\langle \rho_n(u), v \rangle\rangle = 0\right\}$$

come equipped with a free action of the unit quaternions:

$$\mathbb{S}^3 \times E^{10}_n \to E^{10}_n, \quad q \star (u, v) = (qu\bar{q}, qv).$$

Here, $qu\bar{q}$ means that the two quaternionic components of u are simultaneously conjugated by $q \in \mathbb{S}^3$. The quotient of E^{10}_n by the free $\star$-action is a smooth manifold

$$\Sigma^7_n := E^{10}_n / \mathbb{S}^3.$$

For $n = 1$ we have $E^{10}_1 = \mathrm{Sp}(2)$ (the group of quaternionic 2×2 matrices A with $\bar{A}^{\mathrm{t}} A = \mathbb{1}$) and the $\star$-action is the original Gromoll–Meyer action. Hence, $\Sigma^7_1 = \Sigma^7_{\mathrm{GM}}$. It is also easy to see that Σ^7_0 is diffeomorphic to $\mathbb{S}^7$.

Theorem 1. *The differentiable manifold Σ^7_n is a homotopy sphere and represents the $(n \bmod 28)$th element in $\Theta_7 \approx \mathbb{Z}_{28}$.*

Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ denote the diagonal matrices of $\mathrm{O}(2) \subset \mathrm{Sp}(2)$. All E^{10}_n admit a smooth action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^3$ that commutes with the free $\star$-action:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times E^{10}_n \to E^{10}_n, \quad B \bullet (u, v) = (Bu, Bv),$$
$$\mathbb{S}^3 \times E^{10}_n \to E^{10}_n, \quad q \bullet (u, v) = (u, v\bar{q}).$$

The induced effective action on Σ^7_n is an action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$ where $\mathrm{SO}(3) = \mathbb{S}^3/\{\pm 1\}$. On Σ^7_0 this action can be identified with the linear action

$$(B, \pm q) \cdot (x, u) = (Bx, Bqu\bar{q})$$

on $\mathbb{S}^7 \subset \mathbb{R}^2 \times (\operatorname{Im} \mathbb{H})^2$. On $\Sigma^7_1 = \Sigma^7_{\mathrm{GM}}$ the action coincides with the subaction of the $\mathrm{O}(2) \times \mathrm{SO}(3)$-action given in [8].

The surprising fact is the following even/odd grading of the Σ^7_n:

Theorem 2. *All Σ^7_n with even n are equivariantly homeomorphic to $\mathbb{S}^7$ with the linear $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action given above. All Σ^7_n with odd n are equivariantly homeomorphic to the Gromoll–Meyer sphere Σ^7_{GM} with the above $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action. If n is even then all fixed point sets in Σ^7_n are spheres while*