Partial zeta functions

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Abstract. In the present paper, we study analytic properties of the zeta functions defined by the Euler products over elements in subsets of the set of prime elements.


Keywords. Euler product, Analytic continuation, Natural boundary, Dedekind zeta function, Selberg zeta function, Ihara zeta function.

1. Introduction. In our paper [6], we have studied the splitting densities of prime geodesics of negatively curved locally symmetric Riemannian manifolds $X$ in a finite cover $\tilde{X}$ of $X$ as an extension of the prime geodesic theorem. Especially, when the fundamental group of $X$ is $\text{SL}_2(\mathbb{Z})$ and that of $\tilde{X}$ is a congruence subgroup of $\text{SL}_2(\mathbb{Z})$, we have explicitly determined the type of splitting for each geodesic, and calculated the splitting densities for all types. Applying the results in [6] to the formula in [18] about the relation between the Selberg zeta functions for $X$ and $\tilde{X}$, we can obtain an expression of the Selberg zeta function for $\tilde{X}$ as a product over prime geodesics of $X$. By taking the quotients of such expressions of two Selberg zeta functions, we find a formula for the zeta function given by the Euler product over prime geodesics with a certain type of splitting. From this formula, we have obtained the analytic continuation to \{Re $s > 0$\} of such a partial Selberg zeta function (see [6]).

There have been works about partial zeta functions for the Riemann zeta function. In fact, Kurokawa [13] (see also [12] and [14] for the meromorphy of the Euler products in general) already proved that the zeta functions defined by the Euler products over prime numbers $p$ satisfying $(d/p) = 1$ (or $= -1$)
for a fixed square free integer \(d\) are analytically continued to the half plane \(\{\text{Res} > 0\}\) and have natural boundaries on \(\text{Res} = 0\).

The aim of the present paper is to generalize zeta functions defined by partial Euler products and to study their analytic properties with the notations of the Euler products in [12] and [17]. We first get in Theorem 2.1 the analytic continuations in \(\{\text{Res} > 0\}\) by extending the idea used in [13]. Furthermore, in Theorem 2.2, we give the sufficient condition of the distributions of non-trivial zeros for the partial zeta functions having natural boundaries on \(\text{Res} = 0\). As examples, we study the cases of the Dedekind zeta functions for algebraic number fields, the Selberg zeta functions for Riemann surfaces and the Ihara zeta functions for graphs. Actually we show that the partial zeta functions of the Dedekind zeta functions for abelian extensions of \(\mathbb{Q}\), the Selberg zeta functions for congruence subgroups of \(\text{SL}_2(\mathbb{Z})\) and the Ihara zeta functions for finite regular graphs have natural boundaries on \(\text{Res} = 0\).

2. Notations and main results. Let \(P\) be an infinite countable set and \(N : P \to \mathbb{R}_{>1}\) a map such that

\[
\sum_{p \in P} N(p)^{-d} < \infty
\]

for some \(d > 0\). Put \(d_P := \inf\{d > 0 \mid \sum_{p \in P} N(p)^{-d} < \infty\}\) and assume that \(d_P > 0\). For convenience, we normalize \(N\) by \(d_P = 1\). We define the zeta function of \(P\) by

\[
\zeta_P(s) := \prod_{p \in P} (1 - N(p)^{-s})^{-1}, \quad \text{Res} > 1,
\]

and assume that (i) \(\zeta_P(s)\) is non-zero holomorphic in \(\{\text{Res} > 1\}\) and has a simple pole at \(s = 1\), and (ii) \(\zeta_P(s)\) has an analytic continuation to the whole complex plane \(\mathbb{C}\) as a meromorphic function. Note that condition (i) implies the generalized prime number theorem as follows (see [12]).

\[
\#\{p \in P \mid N(p) < x\} \sim \text{li}(x) \quad \text{as} \quad x \to \infty,
\]

where \(\text{li}(x) := \int_2^x (\log t)^{-1} dt \sim x / \log x\).

Let \(G\) be a finite group and \(\hat{G}\) the set of the finite dimensional irreducible unitary representations of \(G\). For a map \(\varphi : P \to \text{Conj}(G)\) and \(\rho \in \hat{G}\), we define the \(L\)-functions by

\[
L_P^{(G)}(s, \rho) = L_P(s, \rho) := \prod_{p \in P} \det (1 - \rho(\varphi(p))N(p)^{-s})^{-1}, \quad \text{Res} > 1,
\]

and assume that, if \(\rho \neq 1\), then \(L_P(s, \rho)\) satisfies that (i') \(L_P(s, \rho)\) is non-zero holomorphic in \(\{\text{Res} \geq 1\}\) and condition (ii) for \(\zeta_P(s)\). Note that condition (i') implies the following generalization of the Tchebotarev-type prime number theorem (see [12] and [17]).

\[
\#\{p \in P \mid \varphi(p) \in [g], N(p) < x\} \sim \frac{\# [g]}{|G|} \text{li}(x) \quad \text{as} \quad x \to \infty
\]

for a given \(g \in G\), where \([g]\) is the \((G-)\)-conjugacy class of \(g \in G\).