Normal coverings of solvable groups

Eleonora Crestani and Andrea Lucchini

Abstract. For a finite non cyclic group $G$, let $\gamma(G)$ be the smallest integer $k$ such that $G$ contains $k$ proper subgroups $H_1, \ldots, H_k$ with the property that every element of $G$ is contained in $H_i^g$ for some $i \in \{1, \ldots, k\}$ and $g \in G$. We prove that for every $n \geq 2$, there exists a finite solvable group $G$ with $\gamma(G) = n$.

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1. Introduction. Let $G$ be a finite group. If $G$ is non cyclic, then $G$ can be obtained as a union of proper subgroups. A covering of $G$ is a set of proper subgroups of $G$ whose union is $G$. The smallest integer $m$ such that $G$ has a covering of cardinality $m$ is denoted $\sigma(G)$. The behaviour of $\sigma(G)$ has been investigated by several authors (see for example [1,3,4,7–11,13]).

More recently, other authors focused their attention on a particular kind of coverings, called normal coverings [2,5,6]. A normal covering for a finite (non cyclic) group $G$ is a covering which is invariant under $G$-conjugation. If $H_1, \ldots, H_k$ are pairwise non-conjugate proper subgroups of $G$ such that $G = \bigcup_{1 \leq i \leq k, g \in G} H_i^g$, then we say that $\{H_1, \ldots, H_k\}$ is a basic set for $G$. The smallest size of a basic set is denoted $\gamma(G)$.

Let $\Sigma$ be the set of the positive integers $n$ with the property that there exists a finite group $G$ with $\sigma(G) = n$. No group is the union of two proper subgroups, so the smallest element of $\Sigma$ is $3 = \sigma(C_2 \times C_2)$. In [13] Tomkinson proved that $7 \not\in \Sigma$. More recently it has been proved that $11, 19, 21, 22, 25 \notin \Sigma$ (see [9]), and this leads to the conjecture that there are infinitely many positive integers outside $\Sigma$. The possible values of $\sigma(G)$ when $G$ is solvable have been characterized by Tomkinson [13]: there exists a finite solvable group $G$ with $\sigma(G) = n$ if and only if $n – 1$ is a prime power.

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In this paper we prove that the behaviour of the function $\gamma$ is quite different. No group is the union of the conjugates of a proper subgroup, so $\gamma(G) > 1$. Unexpectedly, this is the only restriction on the set of possible values of $\gamma$. Indeed we prove:

**Theorem 1.1.** For each positive integer $n \geq 2$, there exists a finite solvable group $G$ with $\gamma(G) = n$.

This is a consequence of the following result. Let $p$ be a prime and let $H_m = C_p \ltimes \cdots \ltimes C_p$ be the iterated wreath product of $m$ cyclic groups of order $p$. If $q$ is a prime and $p$ divides $q - 1$, then, as it is explained in details in Section 3, $H_m$ can be identify with an imprimitive irreducible subgroup of $\text{GL}(p^{m-1}, q)$ and therefore there exists an affine transitive permutation group of degree $q^{p_{m-1}}$ with point-stabilizers isomorphic with $H_m$. Our main result is that, if $G_m$ is this affine transitive group, then $\gamma(G_m) = \min\{m + 1, p + 1\}$. This immediately implies Theorem 1.1: it suffices to notice that, by the Dirichlet’s Prime Number Theorem, for each $m$ there exist two primes $p$ and $q$ with $p \geq m$ and $q - 1$ divisible by $p$.

**2. Preliminary results.** In this section we prove two results concerning the set of elements in a Sylow $p$-subgroup of $\text{Sym}(p^n)$ that are not $p^n$-cycles.

**Lemma 2.1.** Let $P$ be a Sylow $p$-subgroup of the symmetric group $\text{Sym}(p^n)$, and let $\Omega = \{x \in P \mid |x| \neq p^n\}$. There exist $n$ maximal subgroups $M_1, \ldots, M_n$ of $P$ with the property that $\Omega \subseteq \bigcup_{1 \leq i \leq n} M_i$.

**Proof.** The proof is by induction on $n$. If $n = 1$ then $\Omega = \{1\}$ and the statement is trivially true. So we may assume $n > 1$. In this case $P = C \wr Q$ is the wreath product of a Sylow $p$-subgroup $C$ of $\text{Sym}(p)$ with a Sylow $p$-subgroup $Q$ of $\text{Sym}(p^{n-1})$ (see for example [12, Theorem 7.27]). Let $B = C_{p^{n-1}}$ be the basis of the wreath product $C \wr Q$ and let

$$A = \{(c_1, \ldots, c_{p^n-1}) \in B \mid c_1 \cdots c_{p^n-1} = 1\}.$$ 

Clearly $A$ is a normal subgroup of $P$. Now consider $\Omega^* = \{y \in Q \mid |y| \neq p^{n-1}\}$. By the inductive hypothesis, there exist $n - 1$ maximal subgroups $K_1, \ldots, K_{n-1}$ of $Q$ with the property that $\Omega^* \subseteq \bigcup_{1 \leq i \leq n-1} K_i$. For $1 \leq i \leq n - 1$, the semidirect product $M_i = BK_i$ is a maximal subgroup of $P$. Another maximal subgroup of $P$ is $M_n = AQ$. Now let $g \in \Omega$. There exist $(c_1, \ldots, c_{p^n-1}) \in B$ and $\sigma \in Q$ with $g = (c_1, \ldots, c_{p^n-1})\sigma$. If $\sigma \in \Omega^*$, then $\sigma \in K_i$ for some $i \in \{1, \ldots, n - 1\}$ and consequently $g \in M_i$. Assume $\sigma \notin \Omega^*$: then $\sigma$ is a $p^{n-1}$-cycle. Since $g \in \Omega$, we must have

$$1 = g^{p^n-1} = (c_1 c_{1\sigma} \cdots c_{1\sigma^{(p^n-1)-1}}; \ldots, c_{p^n-1} c_{p^n-1\sigma} \cdots c_{p^n-1\sigma^{(p^n-1)-1}})$$

$$= (c_1 c_2 \cdots c_{p^n-1}; \ldots, c_1 c_2 \cdots c_{p^n-1}).$$

In particular $(c_1, \ldots, c_{p^n-1}) \in A$ and $g \in M_n$. We have so proved that $\Omega \subseteq \bigcup_{1 \leq i \leq n} M_i$. \[\square\]