Exact multiplicity for the perturbed Q-curvature problem in $\mathbb{R}^N, N \geq 5$

Abhishek Sarkar and S. Prashanth

Abstract. Let $N \geq 5$ and $\mathcal{D}^{2,2}(\mathbb{R}^N)$ denote the closure of $C^\infty_0(\mathbb{R}^N)$ in the norm $\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\Delta u|^2$. Let $K \in C^2(\mathbb{R}^N)$. We consider the following problem for $\varepsilon \geq 0$:

\[
(P_\varepsilon) \begin{cases} 
\text{Find } u \in \mathcal{D}^{2,2}(\mathbb{R}^N) \text{ solving } \\
\Delta^2 u = (1 + \varepsilon K(x))u^{N+4} \\
u > 0
\end{cases} \text{ in } \mathbb{R}^N.
\]

We show an exact multiplicity result for $(P_\varepsilon)$ for all small $\varepsilon > 0$.

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1. Introduction. Let $N \geq 5$, and let $\mathcal{D}^{2,2}(\mathbb{R}^N)$ denote the closure of $C^\infty_0(\mathbb{R}^N)$ in the norm $\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} := (\int_{\mathbb{R}^N} |\Delta u|^2)^{1/2}$. Let $K \in C^2(\mathbb{R}^N)$. We consider the following problem for $\varepsilon \geq 0$:

\[
(P_\varepsilon) \begin{cases} 
\text{Find } u \in \mathcal{D}^{2,2}(\mathbb{R}^N) \text{ solving } \\
\Delta^2 u = (1 + \varepsilon K(x))u^{N+4} \\
u > 0
\end{cases} \text{ in } \mathbb{R}^N.
\]

We are interested in showing an exact multiplicity result for $(P_\varepsilon)$ for all small $\varepsilon > 0$ (see Theorem 1.5 below).

The above problem is a “perturbed” version of the well-known $Q$-curvature problem which arises in differential geometry. More precisely, the problem is to find out if a given smooth function $Q$ on the $N$-dimensional unit sphere $S^N$ is the $Q$-curvature function of a metric $g$ on $S^N$ which is conformal to the...
standard metric $g_0$. This gives rise to the following problem:

$$(P) \begin{cases} \frac{\Delta^2 g_0}{v} - c_N \Delta g_0 v + d_N v = \frac{N-4}{2} Q v \frac{N+4}{N-4} \\ v > 0 \\ c_N := \frac{1}{2} (N^2 - 2 N - 4), d_N := \frac{1}{16} N (N - 4) (N^2 - 4). \end{cases}$$

The above problem has been studied extensively using the background of differential geometry; see the works [1,3,5] for the geometric context and references to other related works.

We now assume that $Q$ is a perturbation of the constant, viz, $Q = 1 + \varepsilon \tilde{K}$ for a smooth function $\tilde{K}$ on $S^N$ and $\varepsilon > 0$ small. Then, applying the standard stereographic projection from $S^N$ to $\mathbb{R}^N$ on $\tilde{K}$ and $v$ (and calling them $K$ and $u$ respectively), it can be checked that $(P)$ is transformed to $(P_\varepsilon)$.

Existence of solutions to $(P_\varepsilon)$ was done in [3] using variational methods and finite dimensional reduction techniques. To describe their result, we make the following assumptions on $K$:

(K1) $K \in C^2(\mathbb{R}^N)$, \( \|K\|_{L^\infty(\mathbb{R}^N)} + \|\nabla K\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} + \|D^2 K\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N \times \mathbb{R}^N)} < \infty. \)

(K2) (a) There exists $\rho > 0$ such that $\langle \nabla K(x), x \rangle < 0 \ \forall |x| \geq \rho,$
(b) $\langle \nabla K(x), x \rangle \in L^1(\mathbb{R}^N), \int_{\mathbb{R}^N} \langle \nabla K(x), x \rangle dx < 0.$

(K3) The set of all critical points of $K$, denoted by $\text{crit} (K)$, is finite.

(K4) \( \forall \xi \in \text{crit} (K), \) there exists $\beta = \beta_\xi \in (1, N)$ and $a_j \in C(\mathbb{R}^N), 1 \leq j \leq N,$ such that $A_\xi := \Sigma_j a_j(\xi) \neq 0.$ Furthermore, $K(y) = K(\eta) + \Sigma_j a_j|y - \eta|^\beta + o(|x - y|^\beta)$ as $y \to \eta$ for any $\eta$ in a small neighbourhood of $\xi$.

(K5) $\Sigma_{A_\xi < 0} \deg_{\text{loc}}(\nabla K, \xi) \neq (-1)^N.$

From [4], we also recall the following classification results for solutions of $(P_0)$ as well as its linearisation:

**Theorem 1.1.**

(i) Solutions of $(P_0)$ form an $(N + 1)$-dimensional manifold given by

$$\mathcal{M} = \{ z_{\mu, \xi}(x) \overset{\text{def}}{=} C_N \mu \frac{N-4}{2} \left( \mu^2 + |x - \xi|^2 \right)^{\frac{4-N}{2}} : (\mu, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N \}, \quad (1.1)$$

where

$$C_N := [N(N^2 - 4)(N - 4)]^{\frac{N-4}{8}}. \quad (1.1)$$

(ii) (“non-degeneracy”) Solutions of the linearisation of $(P_0)$ form an $(N+1)$-dimensional vector space spanned by $\frac{\partial z_{\mu, \xi}}{\partial \mu}, \frac{\partial z_{\mu, \xi}}{\partial x_1}, \ldots, \frac{\partial z_{\mu, \xi}}{\partial x_N}.$

**Definition 1.2.** Given a set $A \subset \mathbb{R}^+ \times \mathbb{R}^N$, define $\mathcal{M}_A = \{ z_{\mu, \xi} : (\mu, \xi) \in A \}$.

Define the functions $\psi^{(i)}_{\mu, \xi}$ as follows:

$$\psi^{(0)}_{\mu, \xi} = \frac{\partial z_{\mu, \xi}}{\partial \mu}, \quad \psi^{(i)}_{\mu, \xi} = \frac{\partial z_{\mu, \xi}}{\partial x_i}, \quad i = 1, 2, \ldots, N. \quad (1.2)$$