Maximal regularity for non-autonomous evolution equations governed by forms having less regularity

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Abstract. We consider the maximal regularity problem for non-autonomous evolution equations

\[ u'(t) + A(t)u(t) = f(t), \quad t \in (0, \tau] \]
\[ u(0) = u_0. \] (0.1)

Each operator $A(t)$ is associated with a sesquilinear form $a(t)$ on a Hilbert space $H$. We assume that these forms all have the same domain $V$. It is proved in Haak and Ouhabaz (Math Ann, doi:10.1007/s00208-015-1199-7, 2015) that if the forms have some regularity with respect to $t$ (e.g., piecewise $\alpha$-Hölder continuous for some $\alpha > 1/2$) then the above problem has maximal $L_p$-regularity for all $u_0$ in the real-interpolation space $(H, \mathcal{G}(A(0)))_{1-1/p,p}$. In this paper we prove that the regularity required there can be improved for a class of sesquilinear forms. The forms considered here are such that the difference $a(t; \cdot, \cdot) - a(s; \cdot, \cdot)$ is continuous on a larger space than the common domain $V$. We give three examples which illustrate our results.

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1. Introduction and main results. Let $H$ and $V$ be real or complex Hilbert spaces such that $V$ is densely and continuously embedded in $H$. We denote by $V'$ the (anti-)dual of $V$ and by $[\cdot | \cdot]_H$ the scalar product of $H$ and $\langle \cdot, \cdot \rangle$ the duality pairing $V' \times V$. The latter satisfies (as usual) $\langle v, h \rangle = [v | h]_H$ whenever $v \in H$ and $h \in V$. By the standard identification of $H$ with $H'$, we then obtain continuous and dense embeddings $V \hookrightarrow H \approx H' \hookleftarrow V'$. We denote by $\| \cdot \|_V$ The research of the author was partially supported by the ANR project HAB, ANR-12-BS01-0013-02.
and \( \| \cdot \|_H \) the norms of \( V \) and \( H \), respectively. We shall always assume that \( H \) is separable.

We consider the non-autonomous evolution equation
\[
\begin{cases}
u'(t) + A(t) u(t) = f(t), & t \in (0, \tau] \\
u(0) = u_0,
\end{cases}
\] (P)
where each operator \( A(t) \), \( t \in [0, \tau] \), is associated with a sesquilinear form \( \mathfrak{a}(t) \). We assume that \( t \mapsto \mathfrak{a}(t; u, v) \) is measurable for all \( u, v \in V \) and

[H1] \( \text{(constant form domain)} \) \( \mathcal{D}(\mathfrak{a}(t)) = V \).

[H2] \( \text{(uniform boundedness)} \) there exists \( M > 0 \) such that for all \( t \in [0, \tau] \) and \( u, v \in V \), we have \( |\mathfrak{a}(t; u, v)| \leq M \| u \|_V \| v \|_V \).

[H3] \( \text{(uniform quasi-coercivity)} \) there exist \( \alpha_1 > 0, \delta \in \mathbb{R} \) such that for all \( t \in [0, \tau] \) and all \( u, v \in V \) we have \( \alpha_1 \| u \|_V^2 \leq \mathfrak{a}(t; u, u) + \delta \| u \|_H^2 \).

For each \( t \), we can associate with the form \( \mathfrak{a}(t; \cdot, \cdot) \) an operator \( A(t) \) defined as follows
\[
\mathcal{D}(A(t)) = \{ u \in V, \exists v \in H : \mathfrak{a}(t, u, \varphi) = [v | \varphi]_H \forall \varphi \in V \}
\]
\[
A(t)u := v.
\]

On the other hand, there exists a linear operator \( A(t) : V \to V' \) such that \( \mathfrak{a}(t; u, v) = \langle A(t)u, v \rangle \) for all \( u, v \in V \). The operator \( A(t) \) can be seen as an unbounded operator on \( V' \) with domain \( V \) and \( A(t) \) is the part of \( A(t) \) on \( H \), that is,
\[
\mathcal{D}(A(t)) = \{ u \in V, A(t)u \in H \}, \quad A(t)u = A(t)u.
\]

It is a known fact that \(-A(t)\) and \(-A(t)\) both generate holomorphic semigroups \((e^{-s A(t)})_{s \geq 0}\) and \((e^{-s A(t)})_{s \geq 0}\) on \( H \) and \( V' \), respectively. For each \( s \geq 0 \), \( e^{-s A(t)} \) is the restriction of \( e^{-s A(t)} \) to \( H \). For all this, we refer to Ouhabaz [10, Chapter 1].

The notion of maximal \( L_p \)-regularity for the above Cauchy problem is defined as follows.

**Definition 1.1.** Fix \( u_0 \in H \). We say that (P) has maximal \( L_p \)-regularity (in \( H \)) if for each \( f \in L_p(0, \tau; H) \) there exists a unique \( u \in W^1_p(0, \tau; H) \) such that \( u(t) \in \mathcal{D}(A(t)) \) for almost all \( t \), which satisfies (P) in the \( L_p \)-sense.

Recall that under the assumptions [H1]–[H3], J.L. Lions proved maximal \( L_2 \)-regularity in \( V' \) for all initial data \( u_0 \in H \), see e.g. [8], [12, page 112]. This means that for every \( u_0 \in H \) and \( f \in L_2(0, \tau; V') \), the equation
\[
\begin{cases}
u'(t) + A(t) u(t) = f(t) \\
u(0) = u_0
\end{cases}
\] (P')
has a unique solution \( u \in W^1_2(0, \tau; V') \cap L_2(0, \tau; V) \). It is a remarkable fact that Lions’s theorem does not require any regularity assumption (with respect to \( t \)) on the sesquilinear forms apart from measurability. Note however that maximal regularity in \( H \) differs considerably from maximal regularity in \( V' \). The fact that the forms have the same domain means that the operators \( A(t) \) have