Bloom’s inequality: commutators in a two-weight setting

IRINA HOLMES, MICHAEL T. LACEY, AND BRETT D. WICK

Abstract. In 1985, Bloom characterized the boundedness of the commutator \([b, H]\) as a map between a pair of weighted \(L^p\) spaces, where both weights are in \(A_p\). The characterization is in terms of a novel \(BMO\) condition. We give a ‘modern’ proof of this result, in the case of \(p = 2\). In a subsequent paper, this argument will be used to generalize Bloom’s result to all Calderón–Zygmund operators and dimensions.

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1. Introduction and statement of main results. Let \(\mu\) be a weight on \(\mathbb{R}\), i.e. a function that is positive almost everywhere and is locally integrable. Then define \(L^2(\mathbb{R}; \mu) \equiv L^2(\mu)\) to be the space of functions which are square integrable with respect to the measure \(\mu(x)dx\), namely

\[
\|f\|^2_{L^2(\mu)} \equiv \int_{\mathbb{R}} |f(x)|^2 \mu(x)dx.
\]

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For an interval $I$, let $\langle \mu \rangle_I \equiv \frac{1}{|I|} \int_I \mu(x) dx$. And, similarly, set $E^I_I(g) \equiv \frac{1}{\mu(I)} \int_I g \mu dx$.

In [1] Bloom considers the behavior of the commutator
\[ [b, H] : L^p(\lambda) \hookrightarrow L^p(\mu), \]
where $H$ is the Hilbert transform. When the weights $\mu = \lambda \in A_2$, then it is well-known that boundedness is characterized by $b \in BMO$. Bloom however works in the setting of $\mu \neq \lambda \in A_2$, finding a characterization in terms of a $BMO$ space adapted to the weight $\rho = (\frac{\mu}{\lambda})^{\frac{1}{p}}$, namely
\[ \|b\|_{BMO_\rho} \equiv \sup_I \left( \frac{1}{\rho(I)} \int_I |b(x) - \langle b \rangle_I|^2 dx \right)^{\frac{1}{2}}. \]

Recall that $\lambda \in A_p$ if and only if the supremum over intervals below is finite.

\[ [\lambda]_{A_p} = \sup_I \langle \lambda \rangle_I (\lambda^{1-p'})^{p-1} < \infty. \]

**Theorem 1.1** (Bloom [1, Theorem 4.2]). Let $1 < p < \infty$, $\mu, \lambda \in A_p$. Set $\rho = (\frac{\mu}{\lambda})^{\frac{1}{p}}$. Then,
\[ \|[b, H] : L^p(\mu) \to L^p(\lambda)\| \approx \|b\|_{BMO_\rho}. \]

The space $BMO_\rho = BMO$ when $\mu = \lambda$, and this case is well-known. But, the general case is rather delicate, as there are three independent objects in the commutator, the two weights and the symbol $b$. It is remarkable that there is a single condition involving all three which characterizes the boundedness of the commutator.

Commutator estimates are interesting in that operator bounds are characterized in terms of function classes. They generalize Hankel operators, encode weak-factorization results for the Hardy space, and can be used to derive div-curl estimates. Bloom himself applied his inequality to matrix weights. As far as we know, many of these topics remain unexplored in the setting of Bloom’s inequality, and we hope to return to these topics in future papers.

Weighted estimates for commutators are complicated, since $[b, H]$ is essentially the composition of $H$ with paraproduct operators, see (4.1) below. This makes two weight estimates for commutators very difficult. But, the key assumption of both weights being in $A_2$ allows several proof strategies that are not available in the general two weight case. A key property is the ‘joint $A_\infty$ property’, namely that one can quantitatively control Carleson sequences of intervals in both measures. Bloom’s argument is based upon interesting sharp function inequality for the upper bound, and involves an ad hoc argument in the lower bound.

We give an alternate proof of Theorem 1.1 in the case when $p = 2$. This allows us to present the key ideas for a more general result. There are different equivalent formulations of Bloom’s $BMO_\rho$ space, two of which are detailed in Section 2. These formulations are ideal for characterizing certain two weight inequalities for paraproducts in Section 3. Then, $[b, H]$ is a linear combination