Around Schwenninger and Zwart’s zero-two law for cosine families

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Abstract. It is proved that for a cosine family \( \{ c(t) \}_{t \in \mathbb{R}} \) in a normed algebra with a unity \( e \), the following assertions hold: (1) If sup \( t \in \mathbb{R} \| c(t) - e \| < 2 \), then \( c(t) = e \) for every \( t \in \mathbb{R} \). (2) If lim sup \( t \to 0 \| c(t) - e \| < 2 \), then \( \lim_{t \to 0} c(t) = e \). It is also shown that the two respective results, each specific for one of the assertions, are equivalent.

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1. Introduction. Recently, Schwenninger and Zwart [28] proved the zero-two law for cosine families asserting that if \( \{ C(t) \}_{t \in \mathbb{R}} \) is a strongly continuous cosine family on a Banach space \( X \) such that
\[
\limsup_{t \to 0} \| C(t) - I_X \| < 2,
\]
then \( \lim_{t \to 0} \| C(t) - I_X \| = 0 \). (1.2)
The theorem can equivalently be rephrased as saying that the infinitesimal generator of a strongly continuous cosine family \( \{ C(t) \}_{t \in \mathbb{R}} \) satisfying (1.1) is a bounded operator. Schwenninger and Zwart’s result is a generalisation of an earlier result of Fackler [16] which ensures that (1.2) holds under the additional assumption that the cosine family \( \{ C(t) \}_{t \in \mathbb{R}} \) can be represented as
\[
C(t) = \frac{1}{2} (G(t) + G(-t)) \quad (t \in \mathbb{R}),
\]
where \( \{ G(t) \}_{t \in \mathbb{R}} \) is a strongly continuous group on \( X \); this assumption is automatically satisfied when \( X \) has the UMD property [18], and in particular when \( X \) is a Hilbert space. Schwenninger and Zwart’s result also partially generalises...
Arendt’s $0$–$3/2$ law for cosine functions [3] asserting that if \( \{C(t)\}_{t \in \mathbb{R}} \) is a cosine family on a Banach space \( X \), not necessarily strongly continuous, such that
\[
\limsup_{t \to 0} \|C(t) - I_X\| < 3/2,
\]
then
\[
\lim_{t \to 0} \|C(t) - I_X\| = 0.
\]
The zero-two law for cosine functions has had its earlier counterparts for semi-groups and groups of operators on a Banach space, namely as zero-one laws for semi-groups [4, Part II, Lemma 3.1], [14], [29, Remark 3.1.4], and zero-two laws for groups [8,13,16,22,24,26], [27, Section 2.4, Corollary 4.13], respectively.

Schwenninger and Zwart used their zero-two law to prove the following isolability result: if \( \{C(t)\}_{t \in \mathbb{R}} \) is a strongly continuous cosine family on a Banach space \( X \) such that
\[
\sup_{t \in \mathbb{R}} \|C(t) - I_X\| < 2,
\]
then \( C(t) = I_X \) for each \( t \in \mathbb{R} \). This result partially generalises the theorem stating that if \( \{C(t)\}_{t \in \mathbb{R}} \) is a cosine family on a Banach space \( X \), not necessarily strongly continuous, such that
\[
\sup_{t \in \mathbb{R}} \|C(t) - I_X\| < 3/2,
\]
then \( C(t) = I_X \) for each \( t \in \mathbb{R} \). The theorem has an elementary proof based on a straightforward adaptation of the proof of Arendt’s $0$–$3/2$ law for cosine functions, and generalises the result of Bobrowski and Chojnacki [5] stating that if a strongly continuous cosine family \( \{C(t)\}_{t \in \mathbb{R}} \) satisfies
\[
\sup_{t \in \mathbb{R}} \|C(t) - I_X\| < 1,
\]
then \( C(t) = I_X \) for each \( t \in \mathbb{R} \), and the companion result of Chojnacki [9] giving the same conclusion without the need to assume that the cosine family is strongly continuous.

Schwenninger and Zwart’s isolability result and the above-mentioned related results all imply that the identity cosine family on a Banach space \( X \), defined by \( C(t) = I_X \) for each \( t \in \mathbb{R} \), is an isolated point in the space of bounded strongly continuous cosine families on \( X \) when this set is endowed with the metric of uniform convergence corresponding to the operator norm on \( \mathcal{L}(X) \). Here, of course, \( \mathcal{L}(X) \) denotes the Banach space of all bounded linear operators on \( X \). The latter result is a special case of a more general result stating that the so-called scalar cosine families (which, by definition, are the cosine families comprised entirely of scalar multiples of the identity operator) are isolated points in the space of bounded strongly continuous cosine families on a Banach space [5]. A strong quantitative form of this result asserts that if \( a \in \mathbb{R} \) and \( C = \{C(t)\}_{t \in \mathbb{R}} \) is a cosine family on a Banach space \( X \) such that
\[
\sup_{t \in \mathbb{R}} \|C(t) - (\cos at)I_X\| < \frac{8}{3\sqrt{3}},
\]