The non-autonomous Kato class

By

ROLAND SCHNAUBELT (*) and JÜRGEN VOIGT (1)

Abstract. We discuss singular time dependent absorption-excitation rates for second order parabolic equations. A suitable class, the non-autonomous Kato class, is defined for the heat equation in connection with the non-autonomous Miyadera perturbation theorem. We give sufficient conditions in terms of integrability for a potential to belong to the non-autonomous Kato class.

1. Introduction. The Kato class $K$ of potentials $V_0 : \mathbb{R}^n \to \mathbb{R}$ was introduced in 1982 by M. Aizenman and B. Simon, [1], [10], to study Schrödinger operators $-(\Delta + V_0)$ for singular $V_0$. It is a special case of a Stummel class defined in terms of the fundamental solution of the Laplacian $\Delta$ whose importance for second order elliptic operators was realized by T. Kato in [5]. Aizenman and Simon proved that $K$ coincides with the set of potentials having relative bound 0 with respect to $\Delta$ on $L^1(\mathbb{R}^n)$, [1, Thm. 4.5]. Moreover, $K$ is closely related to the set of potentials which are Miyadera perturbations of the semigroup $(U(t))_{t \geq 0}$ on $L^1(\mathbb{R}^n)$ generated by $\Delta$, see [6, 11, 12, 13]. More precisely, the enlarged Kato class is defined by

$$\hat{K} := \{V_0 : \mathbb{R}^n \to \mathbb{C} : V_0 \text{ measurable, } \|V_0\|_U < \infty \},$$

where

$$\|V_0\|_U := \sup_{f \in D(\Delta), \|f\|_1 \leq 1} \int_0^1 \|V_0 U(t)f\|_1 dt.$$ 

(To be precise, in [6, 12, 13] the class $\hat{K}$ was introduced for real potentials $V_0$. However, the results of these papers which are used below carry over to complex-valued $V_0$ by considering $|V_0|$.) We remark that $(\hat{K}, \|\cdot\|_U)$ is a Banach space, see e.g. [13]. For $V_0 \in \hat{K}$ and $\alpha > 0$, let

$$c'_\alpha(V_0) := \sup_{f \in D(\Delta), \|f\|_1 \leq 1} \int_0^\alpha \|V_0 U(t)f\|_1 dt$$

and

$$c(V_0) := \lim_{\alpha \to 0} c'_\alpha(V_0) = \inf_{\alpha > 0} c'_\alpha(V_0),$$

cf. [6, 13]. Then the Kato class is given by

$$K = \{V_0 \in \hat{K} : c(V_0) = 0\},$$

[12, Prop. 4.7, 5.1]. In particular, if $c(V_0) < 1$ then $\Delta + V_0$ with domain $D(\Delta)$ generates a
strongly continuous semigroup on $X = L^1(\mathbb{R}^n)$ by virtue of the Miyadera perturbation theorem, see e.g. [11].

In the present note, we investigate perturbations of the Laplacian by time dependent potentials

$$V : [0, \infty) \times \mathbb{R}^n \to \mathbb{C}.$$ 

By $V(t)$ we denote the multiplication operator with maximal domain induced by $V(t, \cdot)$ on $L^1(\mathbb{R}^n)$. As in the autonomous case, we set

$$\|V(\cdot)\|_U := \sup_{s \geq 0} \sup_{f \in D(\mathcal{A}), \|f\|_1 \leq 1} \frac{1}{t} \int_0^t \|V(s + t)U(t)f\|_1 \, dt,$$

and define the non-autonomous enlarged Kato class by

$$\hat{\mathcal{K}} = \{V : [0, \infty) \times \mathbb{R}^n \to \mathbb{C} : V \text{ measurable, } \|V(\cdot)\|_U < \infty\}.$$ 

Observe that $\|V(\cdot)\|_U = \|V_0\|_U$ for a time independent potential $V(\cdot) = V_0$. We further introduce

$$c'_a(V(\cdot)) := \sup_{s \geq 0} \sup_{f \in D(\mathcal{A}), \|f\|_1 \leq 1} \frac{1}{t} \int_0^t \|V(s + t)U(t)f\|_1 \, dt$$

and

$$c(V(\cdot)) := \lim_{a \to 0} c'_a(V(\cdot)) = \inf_{a > 0} c'_a(V(\cdot)),$$

for $V \in \hat{\mathcal{K}}$ and $a > 0$. We remark that, for $V \in \hat{\mathcal{K}}$, the quantity $c'_a(V(\cdot))$ does not change if we replace “$f \in D(\mathcal{A})$” by “$f \in L^1(\mathbb{R}^n)$” in the second supremum, see [8, Thm. 3.4].

Recently, the Miyadera perturbation theorem has been extended to the non-autonomous situation in [7] and [8]. This result is applied in Theorem 1 to obtain mild solutions of the parabolic problem

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) &= \sum_{k,l=1}^n \frac{\partial}{\partial x_k} a_{kl}(t,x) \frac{\partial}{\partial x_l} u(t, x) + V(t, x)u(t, x), \\
\quad u(s, x) &= f(x), \quad t \geq s, \ x \in \mathbb{R}^n,
\end{aligned}
\end{equation}

provided that $V$ satisfies an estimate similar to $c(V(\cdot)) < 1$. In the remainder of the paper we investigate the class $\hat{\mathcal{K}}$. In Theorem 2 we see that the estimate $c(V(\cdot)) < q$ is equivalent to an integrability condition on $V$ which involves the heat kernel

$$K_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \quad (t > 0, \ x \in \mathbb{R}^n).$$

Recall that $U(t)f = K_t * f$. Further, it is shown in Theorem 6 that Hölder continuity and boundedness of $t \mapsto V(t) \in \hat{\mathcal{K}}$ imply $V \in \hat{\mathcal{K}}$. By means of examples, we first see that continuity is not sufficient for this conclusion and that, second, boundedness in $\hat{\mathcal{K}}$ is not necessary for $V \in \hat{\mathcal{K}}$.

2. Results. First, we study the problem (1) in the space $L^1(\mathbb{R}^n)$, where we assume that

\begin{itemize}
\item [(A)] $a_{kl} \in L^\infty([0, \infty) \times \mathbb{R}^n, \mathbb{R})$ and $\sum_{k,l=1}^n a_{kl}(t,x)\eta_k\eta_l \geq \mu |\eta|^2$ for $x, \eta \in \mathbb{R}^n$, $t \geq 0$, $k, l = 1, \ldots, n$, and a constant $\mu > 0$.
\end{itemize}

It is known that there exists a unique weak fundamental solution

$$\Gamma : \{(t, x, s, y) \in [0, \infty) \times \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n : t \geq s\} \to [0, \infty)$$