An application of a random fixed point theorem to random best approximation

By

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Abstract. The existence of invariant random best approximations in Banach spaces is proved.

1. Introduction and preliminaries. Random fixed point theorems have been used in many instances in approximation theory, see, for example the work of Beg and Shahzad [1, 2], Tan and Yuan [11], Lin [6], and Papageorgiou [8, 9]. In the subject of best approximation, one often wishes to know whether some useful property of the function being approximated is inherited by the approximating function. Meinardus [7] was the first who observed the general principle and employed a fixed point theorem to establish it. The aim of this note is to prove the existence of an invariant random approximation in a Banach space.

Let \((\Omega, A)\) be a measurable space and \(X\) be a metric space. Let \(2^X\) be the family of all nonempty subsets of \(X\) and \(k(X)\) denote the family of all nonempty compact subsets of \(X\). A mapping \(F : \Omega \rightarrow 2^X\) is called \(weakly\ measurable\) (respectively, \(measurable\)) if, for any open (respectively, closed) subset \(C\) of \(X\), \(F^{-1}(C) = \{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in A\). Note that if \(F(\omega) \in k(X)\) for every \(\omega \in \Omega\), then \(F\) is weakly measurable if and only if \(F\) is measurable.

A mapping \(f : \Omega \rightarrow X\) is said to be a \(measurable\ selector\) of a measurable mapping \(F : \Omega \rightarrow 2^X\) if \(f\) is measurable and, for any \(\omega \in \Omega\), \(\xi(\omega) \in F(\omega)\). A mapping \(f : \Omega \times X \rightarrow X\) is called a \(random\ operator\) if, for any \(x \in X\), \(f(\cdot, x)\) is measurable. A measurable mapping \(\xi : \Omega \rightarrow X\) is called a \(random\ fixed\ point\) of a random operator \(f : \Omega \times X \rightarrow X\) if, for any \(\omega \in \Omega\), \(\xi(\omega) = f(\omega, \xi(\omega))\). A random operator \(f : \Omega \times X \rightarrow X\) is \(continuous\) if, for each \(\omega \in \Omega\), \(f(\omega, \cdot)\) is continuous. Two maps \(f, g : X \rightarrow X\) are called \(R\)-\(weakly\ commuting\) if there exists some \(R > 0\) such that \(d(fg(x), gf(x)) \leq Rd(f(x), g(x))\) for all \(x \in X\). Random operators \(f, g : \Omega \times X \rightarrow X\) are \(R\)-weakly commuting if \(f(\omega, \cdot)\) and \(g(\omega, \cdot)\) are \(R\)-weakly commuting for each \(\omega \in \Omega\). We shall make use of the following simplified version of a theorem due to Beg and Shahzad [3].

**Theorem A.** Let \(K\) be a Polish space and \(f, g : \Omega \times K \rightarrow K\) two random operators such that, for each \(\omega \in \Omega\), \(f(\omega, K) \subseteq g(\omega, K)\). If \(f, g\) are \(R\)-weakly commuting, \(g\) is continuous, and

\[
d(f(\omega, x), f(\omega, y)) \leq \lambda d(g(\omega, x), g(\omega, y))
\]


*) Dedicated to Professor Romulus Cristescu on his 70th birthday.
for all \( x, y \in K, \omega \in \Omega \) and some \( \lambda \in (0,1) \) such that \( g(\omega,x) \neq g(\omega,y) \), then \( f \) and \( g \) have a unique common random fixed point.

Let \( S \) be a nonempty subset of a Banach space \( X \). For \( x_* \in X \), let us define
\[
d(x_*,S) = \inf_{y \in S} \|x_* - y\|
\]
and
\[
P_S(x_*) = \{ y \in S : \|x_* - y\| = d(x_*,S) \}.
\]
An element \( y \in P_S(x_*) \) is called a best approximation of \( x_* \) out of \( S \). The set \( P_S(x_*) \) is the set of all best approximations of \( x_* \) out of \( S \). We are now in a position to state and prove our main result.

2. The result.

**Theorem B.** Let \( X \) be a Banach space, \( f, g : \Omega \times X \rightarrow X \) two random operators, and \( S \) a nonempty subset of \( X \) such that \( f(\omega,.) : \partial S \rightarrow S \). Let \( x_* \in X \) and \( x_* = f(\omega,x_*) = g(\omega,x_*) \) for each \( \omega \in \Omega \). Further

\[
(1) \quad \|f(\omega,x) - f(\omega,y)\| \leq \|g(\omega,x) - g(\omega,y)\|
\]
for all \( x, y \in P_S(x_*) \cup \{ x_* \} \) and all \( \omega \in \Omega \). Let \( g(\omega,.) \) be linear and continuous on \( P_S(x_*) \). Suppose \( P_S(x_*) \) is nonempty, compact, starshaped with respect to a point \( q = g(\omega,q) \), and \( g(\omega,P_S(x_*)) = P_S(x_*) \) for each \( \omega \in \Omega \). If

\[
(2) \quad \|f(\omega,g(\omega,x)) - g(\omega,f(\omega,x))\| \leq \frac{R}{k} \|k[f(\omega,x) + (1-k)q] - g(\omega,x)\|
\]
for all \( k \in (0,1) \), \( \omega \in \Omega \), \( x \in P_S(x_*) \), and some \( R > 0 \), then there exists a measurable map \( \xi : \Omega \rightarrow P_S(x_*) \) such that \( \xi(\omega) = f(\omega,\xi(\omega)) = g(\omega,\xi(\omega)) \) for each \( \omega \in \Omega \).

**Proof.** If \( y \in P_S(x_*) \), then \( g(\omega,y) \in P_S(x_*) \) because \( g(\omega,P_S(x_*)) = P_S(x_*) \) for each \( \omega \in \Omega \). It further implies that \( y \in \partial S \) and then \( f(\omega,y) \in S \) because \( f(\omega,\partial S) \subseteq S \) for each \( \omega \in \Omega \). Further, since \( x_* = g(\omega,x_*) = f(\omega,x_*) \), we have from (1) that
\[
\|f(\omega,y) - x_*\| = \|f(\omega,y) - f(\omega,x_*)\|
\]
\[
\leq \|g(\omega,y) - g(\omega,x_*)\|
\]
\[
= \|g(\omega,y) - x_*\|.
\]
Thus \( P_S(x_*) \) is \( f(\omega,.) \)-invariant, that is, \( f(\omega,P_S(x_*)) \subseteq P_S(x_*) \) for each \( \omega \in \Omega \). Choose a sequence \( \{ k_n \} \) of real numbers such that \( 0 < k_n < 1 \) and \( k_n \rightarrow 1 \) as \( n \rightarrow \infty \). For each \( n \), define a random operator \( f_n : \Omega \times P_S(x_*) \rightarrow P_S(x_*) \) as follows:
\[
f_n(\omega,x) = (1 - k_n)q + k_nf(\omega,x).
\]
Obviously, for each \( n \) and each \( \omega \in \Omega \), \( f_n(\omega,.) \) maps \( P_S(x_*) \) into itself since \( P_S(x_*) \) is starshaped with respect to \( q \). Since, for any \( \omega \in \Omega \), \( q = g(\omega,q) \) and \( g(\omega,.) \) is linear on \( P_S(x_*) \), it further implies that
\[
\|f_n(\omega,g(\omega,x)) - g(\omega,f_n(\omega,x))\| = k_n\|f(\omega,g(\omega,x)) - g(\omega,f(\omega,x))\|.
\]