A bound for the least Gaussian prime $\omega$ with $\alpha < \arg(\omega) < \beta$

By

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Abstract. We give an explicit function $B(\theta)$ such that there is a Gaussian prime $\omega$ with $\omega \bar{\omega} < B(\beta - \alpha)$ and $\alpha < \arg(\omega) < \beta$.

1. Introduction. In the present paper, we consider the following problem; for given $(\alpha, \beta)$ with $\alpha < \beta \leq \alpha + \frac{\pi}{2}$, estimate the minimum of norms of Gaussian primes whose arguments are within $(\alpha, \beta)$. (An element $\omega \in \mathbb{Z}[i]$ is called a Gaussian prime if $\omega$ is a prime ideal of $\mathbb{Z}[i]$.) We can give an answer for the problem under “GRH.”

**Theorem 1.** Assume the truth of the Generalized Riemann Hypothesis for $L(s, \psi^r) = \frac{1}{4} \sum_{a} \psi^r((a)) \frac{1}{|a|^{2s}}$ with $\psi^r((a)) = \exp(4ir \arg(a))$ and $r \in \mathbb{Z}$, where $a$ runs over non-zero elements in $\mathbb{Z}[i]$. Then for any real numbers $\alpha, \beta$, with $\alpha < \beta \leq \alpha + \frac{\pi}{2}$, there exists a Gaussian prime $\omega$ with $\alpha < \arg(\omega) < \beta$ such that

$$\omega \bar{\omega} < \frac{A_1}{(\beta - \alpha)} \log \frac{1}{\beta - \alpha},$$

where $A_1$ is a positive absolute constant.

The proof of Theorem 1 employs classical analytic methods for the Hecke $L$-functions with Grössencharacters, using a special integral kernel in [2]. Moreover, we make use of certain trigonometric polynomials in [4], [5], which are majorants or minorants of the characteristic function of interval $(\alpha, \beta)$ on the unit circle.

Next, we consider whether one can say something without GRH.

**Theorem 2.** For any real numbers $\alpha, \beta$, with $\alpha < \beta \leq \alpha + \frac{\pi}{2}$, there exists a Gaussian prime $\omega$ with $\alpha < \arg(\omega) < \beta$ such that

$$\omega \bar{\omega} < \exp \left( \frac{A_2}{\sqrt{\beta - \alpha}} \log^{\frac{3}{2}} \frac{1}{\beta - \alpha} \right),$$

where $A_2$ is a positive absolute constant.

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As a matter of fact, we can get similar results for any imaginary quadratic field (Theorem 3, 4, in the text).

2. Hecke \( L \)-functions and a special integral kernel. First we summarize Hecke’s results which are used in this paper. From now on, we argue about not only the Gaussian field but also imaginary quadratic number fields. Let \( \mathbb{Q}(\sqrt{-d}) \) be an imaginary quadratic field, and \(-d\) its discriminant. Let \( \chi \) be a Grössencharacter of conductor \((1)\) such that

\[
\chi((a)) = \left( \frac{a}{|a|} \right)^u \quad \text{for all} \quad a \in \mathbb{Q}(\sqrt{-d}),
\]

with an integer \( u \). If \( u = 0 \), then \( \chi \) is a character of the ideal class group. We define for a complex number with \( \text{Re}(s) > 1 \)

\[
L(s, \chi) = \sum_I \frac{\chi(I)}{NI^s} = \prod_P (1 - \chi(P)NP^{-s})^{-1},
\]

where \( I \) runs over all integral ideals, \( P \) runs over all prime ideals and \( NI \) is the norm of \( I \). We set

\[
A(s, \chi) = \left\{ (1-s) \right\}^{\delta_\chi} \left( \frac{\sqrt{d}}{2\pi} \right)^s \Gamma \left( s + \frac{|u|}{2} \right) L(s, \chi),
\]

where \( \delta_\chi = 1 \) if \( \chi \equiv 1 \), and 0 otherwise. Hecke showed that \( A(s, \chi) \) is analytically continued to an entire function on the whole \( s \)-plane, and satisfies the functional equation

\[
A(s, \chi) = \frac{1}{\Gamma(1-s)} \frac{|w(\chi)|}{\sigma} A(1-s, \chi^{-1}),
\]

where \( w(\chi) \) is the Gaussian sum of \( \chi \), and \( |w(\chi)| = 1 \).

Our first aim is to compute the following integral in two ways. We set

\[
I_\chi = \frac{1}{2\pi i} \int_{(2)} \frac{L'(s, \chi)}{L(s, \chi)} k(s; x, y) ds
\]

with \( y > x > 1 \), where

\[
\int_{(2)} = \lim_{T \to +\infty} \int_{2-iT}^{2+iT} \quad \text{and}
\]

\[
k(s; x, y) = k(s) = \left( \frac{y^{s-1} - x^{s-1}}{s-1} \right)^2,
\]

which is one of the integral kernels used in [2]. The inverse Mellin transform of \( k(s) \) is given for \( v > 0 \) by

\[
\hat{k}(v; x, y) = \hat{k}(v) = \frac{1}{2\pi i} \int_{(2)} k(s; x, y)v^{-s} ds = \begin{cases} 
0 & \text{if } v \leq x^2, \\
\frac{1}{v} \log \frac{v}{x^2} & \text{if } x^2 \leq v \leq xy, \\
\frac{1}{v} \log \frac{y^2}{v} & \text{if } xy \leq v \leq y^2, \\
0 & \text{if } y^2 \leq v.
\end{cases}
\]