Lp-Bounded Pseudodifferential Operators and Regularity for Multi-quasi-elliptic Equations

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Abstract. Using a classical result of Marcinkiewicz and Lizorkin about the $L^p$-continuity for Fourier multipliers, the authors study the action of a class of pseudodifferential operators with weighted smooth symbol on a family of weighted Sobolev spaces. Results about $L^p$-regularity for multi-quasi-elliptic pseudodifferential operators are also given.

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1. Introduction

Let us fix the attention on the multi-quasi-elliptic operator on the open region $\Omega \subset \mathbb{R}^n$:

$$P(x, D) = \sum_{\alpha \in \mathcal{P}} a_\alpha(x) D^\alpha, \quad a_\alpha(x) \in C^\infty(\Omega), \quad \alpha \in \mathbb{Z}_+^n, \quad D_{x_j} = -i \partial_{x_j},$$

such that for any compact subset $K$ of $\Omega$, using standard vectorial notation,

$$\left| \sum_{\alpha \in \mathcal{F}(\mathcal{P})} a_{\alpha}(x)\xi^{\alpha} \right| > C_K \Lambda_{\mathcal{P}}(\xi) \quad \text{when} \quad x \in K \quad \text{and} \quad |\xi| > C_K, \quad C_K > 0. \quad (1.1)$$

Here $\mathcal{P}$ is a convex polyhedron in $\mathbb{R}_+^n$ with border $\mathcal{F}(\mathcal{P})$ and $\Lambda_{\mathcal{P}}(\xi)$ is a suitable weight function; they will be completely described in the next section.

In the $L^2$ frame the local solvability and regularity, also in microlocal sense, of multi-quasi-elliptic operators are at the moment standard arguments; about them we quote Gindikin-Volevich [[18], Ch. 1, §4], Boggiatto-Buzano-Rodino [[3], Ch. 1, §1.8] and, for wider classes of differential and pseudodifferential operators, Rodino [14], Garello [8], Beals [1] and finally the general Weyl-Hörmander pseudodifferential calculus [[11] §4,5], [[12], Ch. 18].

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Fixing now the attention on the local regularity it is well-known that \( u \in H^{P, \text{loc}}(\Omega) \)
whenever \( P(x, D)u \in L^2_{\text{loc}}(\Omega) \), \( P(x, \xi) \) satisfies (1.1) and \( H^{P, \text{loc}}(\Omega) \) is given by the
distributions \( u \in \mathcal{D}'(\Omega) \) such that \( \Lambda_P(D)(\varphi u)(x) \in L^2(\mathbb{R}^n) \), for any \( \varphi \in C_c^\infty(\Omega) \).

In the present paper we extend such a result to the case \( P(x, D)u \in L^p_{\text{loc}}(\Omega) \), for \( 1 < p < \infty \). More precisely we construct a parametrix of \( P(x, D) \) in the framework
of a new class of \( L^p \)-bounded pseudodifferential operators, suitably defined.

For the multi-quasi-elliptic partial differential equations with constant coefficients
the same result was essentially stated by L. Cattabriga \[5\], 1967.

Using a suitable weight function \( \Lambda(\xi) \), in \( \S 2 \) we introduce the weighted symbol classes \( M^{m_\rho}_\Lambda(\Omega) \), \( m \in \mathbb{R} \), \( \rho = \rho(\Lambda) \in [0, 1] \), which actually are an extension of the classes \( M^{m_\rho}_\rho(\Omega) \) introduced in Taylor [[17] Ch. XI, \( \S 4 \)], 1981.

Emphasis here is given in the choice of the weight function \( \Lambda(\xi) \) which must assure
both the \( L^p \)-continuity of the pseudodifferential operators in \( \text{Op} M^{0}_{\rho, \Lambda}(\Omega) \) and the symbolical calculus; at the same time it should be so general to apply to the study
of regularity of multi-quasi-elliptic operators.

Essentially all the matter is obtained by using the Marcinkiewicz-Lizorkin Lemma
about Fourier multipliers \[13\] 1963, see also Stein [[16], Ch. IV, \( \S 6, \text{Th. 6'} \)]. It
takes here the place of the Miklin-Hörmander Theorem \[10\], which is a basic step
in proving the \( L^p \)-continuity, \( 1 < p < \infty \), of the pseudodifferential operators in \( \text{Op} S_{1,0} \).

In the next section we introduce the weight function classes using in the following.
Particular care is devoted to the description of the functions \( \Lambda_P(\xi) \).

In \( \S 3 \) we define the symbol classes \( M^{m_\rho}_\rho(\Omega) \) both in local and global shape; the pseudodifferential calculus of the related operators is then developed.

The \( L^p \)-continuity of such operators, when \( m = 0 \) also in the framework of the
global symbol classes \( M^{0}_{\rho, \Lambda}(\Omega) \), is stated in \( \S 4 \), while in \( \S 5 \) the weighted Sobolev spaces
are introduced.

In \( \S 6 \) and \( \S 7 \), the parametrix for pseudodifferential operators with weighted elliptic
symbol is constructed and \( L^p \)-regularity results for such operators are given.

2. Weight Functions

In the following we will say that a positive map \( \Lambda(\xi) \in C^\infty(\mathbb{R}^n) \) is a weight function
if it satisfies the following assumptions:

\begin{enumerate}
  \item a. \( \Lambda \) has "polynomial growth", i.e. for some positive constants \( \mu_0 \leq \mu_1 \) and \( c < C \):

\[ c(1 + |\xi|)^{\mu_0} \leq \Lambda(\xi) \leq C(1 + |\xi|)^{\mu_1} \quad \text{for every} \quad \xi \in \mathbb{R}^n; \quad (2.1) \]

  \item b. there exists a positive real constant \( \mu \geq \mu_1 \) such that for any \( \alpha, \gamma \in \mathbb{Z}_+^n \) with \( \gamma_j \in \{0, 1\}, j = 1, 2, ..., n; \)

\[ |\xi^{\gamma} \partial^{\alpha + \gamma} \Lambda(\xi)| \leq C_{\alpha, \gamma} \Lambda(\xi)^{1 - \frac{1}{\rho[\alpha]}} \quad \text{for every} \quad \xi \in \mathbb{R}^n, \quad (2.2) \]

where agreeing with the standard multi-index notation \( \xi^{\gamma} = \xi_1^{\gamma_1} \ldots \xi_n^{\gamma_n} \).
\end{enumerate}