The Quasiapproximate \((\mathcal{U} + \mathcal{K})\)-Invariants of Essentially Normal Operators

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Abstract. For a class of essentially normal operators, we characterize their norm closures of \((\mathcal{U} + \mathcal{K})\)-orbits. Moreover, we introduce a notion of the quasiapproximate \((\mathcal{U} + \mathcal{K})\)-equivalence of essentially normal operators and determine completely the quasiapproximate \((\mathcal{U} + \mathcal{K})\)-invariants. Finally, we give the canonical forms of essentially normal operators under this quasiapproximate \((\mathcal{U} + \mathcal{K})\)-equivalence.

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1. Introduction

Let \(\mathcal{H}\) be a complex separable Hilbert space. \(\mathcal{L}(\mathcal{H})\) denotes the set of bounded linear operators acting on \(\mathcal{H}\) and \(\mathcal{K}(\mathcal{H})\) denotes the ideal of compact operators on \(\mathcal{H}\).

The classification of operators is in a key position in operator theory. First, let us briefly mention some well-known results on this subject.

Given \(T \in \mathcal{L}(\mathcal{H})\), \(\mathcal{U}(T) := \{UTU^* : U \in \mathcal{L}(\mathcal{H}) \text{ a unitary operator}\}\) is called the unitary orbit of \(T\). If \(A \in \mathcal{U}(T)\), we write \(A \cong T\). D. Voiculescu\(^1\) characterized \(\mathcal{U}(T)\), the norm closure of \(\mathcal{U}(T)\). For a normal operator \(N\), it is easy to show that \(A\) is normal if \(A \in \mathcal{U}(N)\). It was known that E. Hellinger’s multiplicity theory determines completely \(\mathcal{U}(N)\). Moreover, using a result of D. Voiculescu, D. W. Hadwin\(^2\) also described the \(\mathcal{U}(N)\) in concrete terms.

One can also turn one’s attention to the Calkin algebra \(\mathcal{A}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})\). An operator \(A \in \mathcal{L}(\mathcal{H})\) is called essentially normal if \(A^*A - AA^* \in \mathcal{K}(\mathcal{H})\) or equivalently, \(\pi(A)\) is a normal element in \(\mathcal{A}(\mathcal{H})\), where \(\pi\) denotes the canonical

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map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{A}(\mathcal{H})$. In what follows, $(\mathcal{E}N)(\mathcal{H})$, or briefly, $(\mathcal{E}N)$ always denotes the set of essentially normal operators acting on $\mathcal{H}$.

**Theorem BDF**[^3]. Let $A, B \in (\mathcal{E}N)(\mathcal{H})$. Then $A$ is unitarily equivalent to some compact perturbation of $B$ if and only if $\sigma_e(A) = \sigma_e(B)$ and $\text{ind} (A-\lambda) = \text{ind} (B-\lambda)$ for all $\lambda \notin \sigma_e(A)$.

Nevertheless, it should be pointed out that essentially unitary equivalence fails to retain some useful information out of the spectral picture. We shall explain index and spectrum of operators in next section.

It follows from a well known Putnam-Fuglede theorem that two normal operators are similar if and only if they are unitarily equivalent. But, for non-normal operators, it is very hard to find a complete set of similarity invariants. Moreover, when $T$ is compact, and where $T$ is an invertible operator of the form unitary plus compact.

If $S \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$ we write $S \simeq_{u+k} T$. $S$ and $T$ are said to be approximately $(\mathcal{U} + \mathcal{K})$–equivalent if $(\mathcal{U} + \mathcal{K})(S) = (\mathcal{U} + \mathcal{K})(T)$. Hence, one pay more attention to approximate similarity. Given $T \in \mathcal{L}(\mathcal{H})$, $S(T) := \{XTX^{-1} : X \in \mathcal{L}(\mathcal{H}) \text{ an invertible operator} \}$ is called the similarity orbit of $T$. If $S \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$ then $\pi(S)$ is unitarily equivalent to $\pi(T)$. Moreover, it is worth to observe that if $T \in (\mathcal{E}N)(\mathcal{H})$, then $(\mathcal{U} + \mathcal{K})(T) \subset (\mathcal{E}N)(\mathcal{H})$. For these reasons, when $T \in (\mathcal{E}N)(\mathcal{H})$ it is rational to consider $(\mathcal{U} + \mathcal{K})(T)$.

In [6], $(\mathcal{U} + \mathcal{K})(T)$ was characterized in the cases where $T$ is normal, where $T$ is compact, and where $T$ is the unilateral (forward) shift. The normal case is particularly elegant because it shows that normal operators can be regarded as the canonical forms of a class of almost normal operators (i.e., can be written as normal plus compact) under approximate $(\mathcal{U} + \mathcal{K})$–equivalence. L. Marcoux[^7] also determined $(\mathcal{U} + \mathcal{K})(T)$, where $T$ is a shift-like. This answers affirmatively a question posed by D. A. Herrero. For a natural number $n$ and an analytic Jordan region $\Omega$, let $A(\Omega, n)$ be the class of those operators $T$ in $(\mathcal{E}N)(\mathcal{H})$ satisfying the following conditions.

1. $\sigma(T) = \overline{\Omega}, \Omega \cap \sigma_e(T) = \emptyset$.
2. $\text{ind} (T-\lambda) = -n$, and $\ker (T-\lambda) = \{0\}, \forall \lambda \in \Omega$.

Motivated by above work, Y. Q. Ji, C. L. Jiang and Z. Y. Wang[^8] gave a spectral