Regular Functions of Operators on Tensor Products of Hilbert Spaces

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Abstract. A class of linear operators on tensor products of Hilbert spaces is considered. Estimates for the norm of operator-valued functions regular on the spectrum are derived. These results are new even in the finite-dimensional case. By virtue of the obtained estimates, we derive stability conditions for semilinear differential equations. Applications of the mentioned results to integro-differential equations are also discussed.

Mathematics Subject Classification (2000). Primary 47A55, 47A75; Secondary 47G10, 47G20.

Keywords. Hilbert spaces; tensor products; operator functions; integro-differential equations.

1. Introduction

Let $E_j$ ($j = 1, 2$) be separable Hilbert spaces with scalar products $\langle \cdot, \cdot \rangle_j$, the unit operators $I_j$, and the norms $\| \cdot \|_j = \sqrt{\langle \cdot, \cdot \rangle_j}$. Let $H = E_1 \otimes E_2$ be the tensor product of $E_1$ and $E_2$. This means that $H$ is a collection of all formal sums of the form

$$u = \sum_j y_j \otimes h_j \quad (y_j \in E_1, h_j \in E_2)$$

with the understanding that

$$\lambda(y \otimes h) = (\lambda y) \otimes h = y \otimes (\lambda h), \quad (y + y_1) \otimes h = y \otimes h + y_1 \otimes h,$$

$$y \otimes (h + h_1) = y \otimes h + y \otimes h_1.$$

Here $y, y_1 \in E_1, h, h_1 \in E_2$, and $\lambda$ is a number.

The scalar product in $H$ is defined by

$$\langle y \otimes h, y_1 \otimes h_1 \rangle_H = \langle y, y_1 \rangle_1 \langle h, h_1 \rangle_2 \quad (y, y_1 \in E_1, h, h_1 \in E_2)$$

This research was supported by the Kamea Fund of the Israel.
and the cross norm \( \| . \|_H := \sqrt{\langle \cdot , \cdot \rangle_H} \), cf. [8, p. 30]. In addition, \( I = I_H \) means the unit operator in \( H \). For a linear operator \( A \), \( \sigma (A) \) is the spectrum, \( \text{Dom} (A) \) is the domain, \( R_\lambda (A) := (A - I\lambda)^{-1} \) is the resolvent, \( \lambda_k (A) (k = 1, 2, \ldots) \) are the eigenvalues with their multiplicities, \( \text{co}(A) \) is the closed convex hull of \( \sigma (A) \), \( A_I = (A - A^*)/2i \) is the Hermitian component. The asterisk means adjointness.

Consider the operator

\[
A = M_1 \otimes I_2 + I_1 \otimes M_2,
\]

where \( M_j \) are linear operators in \( E_j \) \((j = 1, 2)\). Let \( f(z) \) be a scalar-valued function analytic on some neighborhood of \( \sigma (A) \). As usual, we define the function \( f(A) \) of \( A \) by

\[
f(A) := \frac{1}{2\pi i} \int_L f(\lambda) (I\lambda - A)^{-1} d\lambda
\]

provided \( A \) is bounded. Here \( L \) is a closed Jordan contour, surrounding \( \sigma (A) \) and having positive orientation with respect to \( \sigma (A) \).

In the present paper, we derive estimates for the norm of functions of the operator \( A \) defined by (1.1). Our results supplement the well-known results on analytic operator functions, cf. [12]. They are new even in the finite-dimensional case, cf. [2, 8, 11].

The paper is organized as follows. In Sections 2 and 3 we investigate finite-dimensional operators. Section 4 is devoted to the case when \( M_1, M_2 \) are Hilbert-Schmidt operators. In Section 5 we consider bounded quasi-Hermitian operators; that is, bounded operators having compact Hermitian components. In Section 6 the results from Section 5 are generalized to unbounded operators. Section 7 is devoted to the absolute stability of semilinear differential equations in a Hilbert space. The stability of integro-differential equations is discussed in Section 8.

2. Finite dimensional operators

Let \( C^n \) be the \( n \)-dimensional Euclidean space with the Euclidean norm \( \| . \|_n \) and \( B(C^n) \) be the set of all linear operators in \( C^n \). In this section \( E_j = C^{m_j}, H = C^n = C^{m_1} \otimes C^{m_2} \) with \( n = m_1 m_2 \). Introduce the numbers

\[
\gamma_{n,p} := \sum_{1 \leq k_1 < k_2 < \ldots < k_p \leq n-1} 1^{1/2} (n-1)^{-p/2} = \frac{(n-1)!}{(n-p)! p!},
\]

\((p = 1, \ldots, n-1)\) and \( \gamma_{n,0} = 1 \). Here and below

\[
\binom{n}{p} = n!/(n-p)! p!
\]

are the binomial coefficients. Simple calculations show that

\[
\gamma_{n,k} \leq \frac{1}{\sqrt{k!}} \quad (k = 1, \ldots, n-1).
\]

(2.1)