A Variational Principle in Krein Space II

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Abstract. Let $A$ be a self-adjoint operator in a Krein space $(K, [\cdot, \cdot])$. Under certain natural assumptions, it is shown precisely which real eigenvalues of $A$ can be given a max-inf characterization generalizing the usual one in Hilbert space. This result unifies several approaches in the recent literature.

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1. Introduction

We shall study double extremum principles for eigenvalues of a self-adjoint operator $A$ in a Krein space $(K, [\cdot, \cdot])$. In the case where the inner product is positive (so $K$ is a Hilbert space) the topic is over a century old and has been developed in many directions. One result in this area (cf. [20]) states that if the eigenvalues $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_\omega$ ($\omega \leq \infty$) of $A$ below the infimum $M$ of the essential spectrum of $A$ are labelled in increasing order (counted by multiplicity), then the variational quantity

$$\sigma_k := \max \{ \inf \{ [Ax, x]/[x, x] \mid x \in T \cap \mathcal{D}(A), x \neq 0 \} \mid \text{codim } T = k - 1 \}$$

(1.1)

is equal to $\lambda_k$ for $k \leq \omega$ and $\sigma_k = M$ if $k > \omega$.

The corresponding theory for Krein spaces is quite recent, however, and is partly stimulated by generalized eigenvalue problems of the form

$$Cx = \lambda Bx,$$

(1.2)
common in applications, where \( C \) and \( B \) are self-adjoint in a Hilbert space \( (\mathcal{H}, (\cdot, \cdot)) \). If \( B \) is bounded and boundedly invertible, a new (in general indefinite) inner product can be given by \( [x, y] = (x, By) \) leading to a Krein space \( K \). Then (1.2) is equivalent to

\[
Ax = \lambda x,
\]

where \( A := B^{-1}C \) is self-adjoint in \( K \).

Since the finite dimensional situation has already been treated in [6] (which includes a review of the relevant literature), we assume from now on that \( A \) is a self-adjoint operator in an infinite dimensional Krein space \( K \). The spectrum of such an operator can be significantly more complicated than in the Hilbert space case, cf. [2, 10]. For example, it could be the whole complex plane (cf. [11]), and some restrictions are necessary to obtain a meaningful variational principle. For the case where \( A \) is nonnegative, triple extremum principles can be found in [14], [17], and [19]. In general, one needs restrictions on both forms \([Ax, x]\) and \([x, x]\) in order for the inf in (1.1) to take finite values.

Double extremum principles for this situation are fairly recent, and the most general we know is given in [5] which imposes various restrictions on the above forms. One is to allow only positive vectors (i.e., for which \([x, x] > 0\) in (1.1). As a result, only eigenvalues of nonnegative type (i.e., for which the corresponding eigenvectors are nonnegative in \( K \)) are characterized. We shall follow suit, leaving the analogous results for nonpositive type eigenvalues to the reader.

We remark that [5] also contains a review of the relevant literature, much of which (e.g., [1, 8]) is in the setting of (1.2). Here we shall extend the cited variational principles to larger classes of operators and eigenvalues, and significantly improve the applicability of the results. In fact we shall show precisely which eigenvalues can be characterized by such extremum principles. With the aid of recent developments in the theory, for example of critical points which could be regular (e.g., nonsemisimple eigenvalues) or singular (hence embedded in the essential spectrum), we either eliminate the assumptions of [5] or show them to be necessary in certain senses.

For example, another restriction introduced in [5] is that \( A \) should be Translated Quasi-Uniform Positive (TQUP). This means that one can translate the eigenparameter by some real number \( \nu \) so that the corresponding operator \( A - \nu I \) is QUP, i.e., is uniformly positive on some subspace of finite codimension. QUP operators already appeared in different contexts, for example in [15] and [3], and have been studied in their own right in [12], where the terminology was introduced. We shall see in Section 2 and Appendix A that the TQUP assumption is necessary for a nontrivial variational principle of the kind studied here.

The concept of type, mentioned earlier for eigenvalues, extends to the essential spectrum, cf. [2, 15]. For a TQUP operator, we shall see that this part of the spectrum is real and splits into negative type \( \Sigma^- \) with supremum \( m \) and positive type \( \Sigma^+ \) with infimum \( M \), where \( m < M \). The (discrete) spectrum between \( m \) and \( M \) can similarly be split into consecutive groups of eigenvalues of entirely